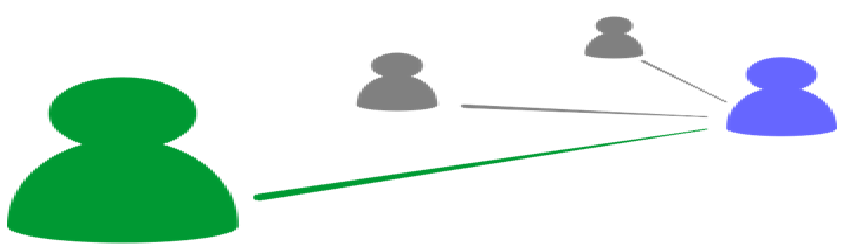
Proceedings for the 48th Annual Meeting of the Research Council on Mathematics Learning

Reestablishing Connections for Mathematics Learning



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Please Note:

RCML History

The Research Council on Mathematics Learning, formerly The Research Council for Diagnostic and Prescriptive Mathematics, grew from a seed planted at a 1974 national conference held at Kent State University. A need for an informational sharing structure in diagnostic, prescriptive, and remedial mathematics was identified by James W. Heddens. A group of invited professional educators convened to explore, discuss, and exchange ideas especially in regard to pupils having difficulty in learning mathematics. It was noted that there was considerable fragmentation and repetition of effort in research on learning deficiencies at all levels of student mathematical development. The discussions centered on how individuals could pool their talents, resources, and research efforts to help develop a body of knowledge. The intent was for teams of researchers to work together in collaborative research focused on solving student difficulties encountered in learning mathematics.

Specific areas identified were:

1. Synthesize innovative approaches.
2. Create insightful diagnostic instruments.
3. Create diagnostic techniques.
4. Develop new and interesting materials.
5. Examine research reporting strategies.

As a professional organization, the **Research Council on Mathematics Learning (RCML)** may be thought of as a vehicle to be used by its membership to accomplish specific goals. There is opportunity for everyone to actively participate in **RCML**. Indeed, such participation is mandatory if **RCML** is to continue to provide a forum for exploration, examination, and professional growth for mathematics educators at all levels.

The Founding Members of the Council are those individuals that presented papers at one of the first three National Remedial Mathematics Conferences held at Kent State University in 1974, 1975, and 1976.

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Reestablishing Connections for Preservice Teacher Beliefs, Perceptions, & Understanding

LOOKING FOR RELATIONSHIPS: INVESTIGATING PRESERVICE TEACHERS' SPATIAL ABILITY, SELF-EFFICACY, AND MATHEMATICS ACHIEVEMENT

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This study explored mathematics self-efficacy (MSE), spatial ability, and achievement for 65 preservice teachers (PST). Findings indicate statistically significant differences between the secondary and elementary PST MSE scores with a moderate effect size on the overall MSE as well as the two subscales for this construct. The study also revealed that the spatial ability, GPA, and ACT scores were significantly higher for the preservice secondary teachers than for the preservice elementary teachers. The results indicate that the overall MSE and spatial ability had a moderate level of correlation. There were also moderate correlations among MSE, spatial ability and ACT scores.

Introduction

Spatial ability/reasoning has a deep and lengthy history in mathematics education research with much of this work exploring the connection of spatial ability with mathematics achievement (Fennema & Sherman, 1977; Uttal et al., 2013; Wang, 2020). While there is some research exploring affective measures and spatial ability, such as math anxiety (Wang, 2020), there are limited studies exploring spatial reasoning with self-efficacy or making comparisons between elementary and secondary preservice teachers. One exception is with a study exploring students' attitudes and spatial ability in conjunction with geography students (Shin et al., 2015). However, that study explored attitudes specific to spatial thinking. Another study explored self-efficacy, but it was specific to geometry (Dursun, 2010). With evidence that spatial ability and self-efficacy are connected to various outcomes and choices (Jo & Bednarz, 2014; Skaalvik & Skaalvik, 2017), it is an important area for further research. The purpose of this study is to explore possible correlations between mathematics self-efficacy, spatial ability, and mathematics achievement for elementary and secondary preservice teachers (PSTs) as well as potential differences between elementary and secondary PSTs.

Background Literature

Mathematics Self-Efficacy

Bandura (2001) defines self-efficacy as an individual's belief in their ability to perform certain tasks. Extending Bandura's work, researchers define mathematics self-efficacy as one's belief that he/she can complete mathematics tasks/problems successfully (Hackett & Betz, 1989; Zuya et al., 2016). Preservice teachers' self-efficacy has been connected with their occupational commitment (Klassen & Chiu, 2011), instructional choices (Haverback, 2009), classroom management (Dussault, 2006), learner-centered beliefs (Dunn & Rakes, 2011), and a variety of other factors (see Zee & Koomen, 2016). These studies highlight the importance of exploring self-efficacy with PSTs as it connects to a variety of outcomes and behaviors.

Spatial Ability

Spatial ability is defined as the ability to manipulate and interact with one's environment through two kinds of lenses: spatial visualization and spatial orientation (Owens, 1990). Spatial visualization is the ability to manipulate mental models through various transformations, which maintains the manipulator place in space while changing the placement of the model. The viewer stays but the object moves. Spatial orientation is the ability to see a mental model from various perspectives while the mental model remains still. In other words, the viewer moves while the object stays (see Harris et al., 2013; Ramey & Uttal, 2017).

Battista (1994) goes further in his description of spatial ability by relating it to other mental processes particularly the cognitive mapping that learners use when learning new material. He argued that learners start with viewing a concept from *multiple perspectives*, then create landmarks for important ideas, and finally establish routes/connections between these landmarks in order to navigate the new conceptual framework. He also stated that these levels closely mirrored Van Hiele levels of geometric thinking, which suggests a vital connection between spatial ability and mathematics. When learners are presented with spatial reasoning problems, they must break down previously held ideas to forge new ones (Uttal et al., 2013).

Research has also shown that there is a positive relationship between self-efficacy related to perceptions of spatial ability and spatial ability (Towle et al., 2005). However, much of this research explored measures other than self-efficacy and was not conducted with PSTs. Research also indicates that preservice teachers' spatial ability is lower than other college majors, which can affect their pedagogical content knowledge and their ability to teach effectively in the classroom (Shin et al., 2015). With limited research on related factors (self-efficacy and achievement) with spatial ability as well as differences between elementary and secondary preservice teachers, our study addresses the following research questions:

1. What are the mathematics self-efficacy, spatial ability and mathematics achievement for secondary and elementary preservice teachers?

2. What is the strength of association between preservice teachers' spatial ability, mathematics self-efficacy, and achievement in mathematics (college math course grades and math ACT score)?

3. Are there any significant differences among the mathematics self-efficacy, spatial ability and mathematics achievement for secondary and elementary preservice teachers?

Method

Participants

Preservice teachers (PSTs) in their last two years at a land grant university in the southwest were surveyed. The 65 participants included 26 secondary science and mathematics PSTs and 39 elementary PSTs. Their mean age was 21.1 years, with 17% males and 83% females.

Instruments

The survey items included demographic items (e.g., gender, year in college, race, and ethnicity), the Purdue Visualization of Rotations (ROT) assessment to measure spatial reasoning, and the Mathematics Self-Efficacy Scale (MSES).

Purdue Visualization of Rotations. The Purdue Visualization of Rotations (ROT) test consists of 20 multiple choice question items and has been shown to be a reliable and valid instrument for assessing spatial ability with preservice teachers (Jo & Bednarz, 2014). The ROT measures the ability to visualize rotations of three-dimensional objects and was revised by Bodner and Guay (1997). The ROT is scored with one point for a correct response and zero for an incorrect response with a possible range of 0 to 20.

Mathematics Self-Efficacy Scale (MSES). The Mathematics Self-Efficacy Scale, developed by Betz and Hackett (1983), contains 34 items identified as relevant to the study of mathematics-related self-efficacy expectations. The two subscales are (1) Mathematics Tasks -18 items and (2) Mathematics Courses -16 mathematics-related college classes. The response format is a ten-point Likert scale ranging from (0) "no confidence at all" to (9) "complete confidence". The total scores were found by adding the responses to all 34 items and then dividing the sum by 34 to get the average. Betz and Hackett (1983) reported an overall high internal consistency for the scale (r = .96), which matched the reliability score in this study (r = .96).

Achievement. Informed consent was granted to obtain the grades from college math courses and math ACT score if available. Grades were used to calculate a grade point average (GPA).

Analysis

To address Research Question 1, means were reported on elementary, secondary, and overall for spatial ability, ACT, GPA and self-efficacy scores for participants. To address Research Question 2, a Pearson correlation coefficient was calculated to determine if spatial ability and ACT scores for PSTs were correlated. A Spearman's rho was performed to determine if spatial ability and mathematics self-efficacy scores or spatial ability and GPA for PSTs were correlated. Effect sizes were also reported with findings. Finally, to address Research Question 3, a Mann Whitney U test, due to homogeneity of variance being violated, was performed to determine if there were significant differences between elementary and secondary PSTs' self-efficacy scores as well as GPA. In addition, an independent t-test was used to assess differences for spatial ability and ACT scores between elementary and secondary PSTs.

Results

Research Question 1

To analyze data collected in the study, the means and standard deviations are reported in Table 1. These were used to examine the data for preservice teachers' mathematics self-efficacy, spatial ability and achievement of preservice teachers (PSTs).

Table 1

Means and Standard Deviations for MSE, SA and Achievement of PSTs

Construct	Secondary (*n*=26) *M (SD)*	Elementary (*n*=39) *M (SD)*	Overall (*n*=65) *M (SD)*
Spatial Ability	13.88 (3.30)	10.95 (3.85)	12.12 (3.90)
Self-Efficacy			
Overall MSE	6.58 (1.10)	5.34 (1.56)	5.84 (1.51)
MT-SE	6.88 (1.12)	5.68 (1.63)	6.16 (1.56)
MRSS-SE	6.25 (1.40)	4.96 (1.69)	5.48(1.69)
Achievement			
ACT	25.54 (3.40)	22.54 (3.83)	23.77 (3.92)
GPA	2.79 (0.86)	3.35 (0.53)	3.13 (0.73)

Note: MT-SE – Mathematics Tasks Self-Efficacy, MRSS-SE – Mathematics Related School Subjects Self-Efficacy

Research Question 2

The Pearson's r and the Spearman's rho were calculated to determine if a possible correlation existed among mathematics self-efficacy, spatial ability, GPA and ACT scores. Spatial ability and ACT scores, $r(63) = .45$, $p < .001$, as well as mathematics self-efficacy and spatial ability were positively correlated, r_s (63) = .39, $p < .01$. Mathematics self-efficacy and ACT scores was positively correlated, r_s (63) = .58, $p < .01$.

Research Question 3

A Mann-Whitney U test indicated that the overall mathematics self-efficacy of the secondary PSTs (Mdn = 6.78) was significantly greater than that for the elementary PSTs (Mdn = 5.21), U = 259.00, $p < .01$ as well as the GPA of secondary PSTs (Mdn = 2.78) versus elementary PSTs (Mdn = 3.25), U = 313.50, $p < .01$ (see Table 2).

Table 2

Mann-Whitney for MSE and GPA of PSTs

Construct	Secondary (*n*=26) *Mdn*	Elementary (*n*=39) *Mdn*	*p*	*r*
Overall MSE	6.78	5.21	*p* < .01**	-.41
GPA	2.78	3.25	*p* < .01**	-.32

Note: ** = p < .01, Mdn = median

An independent samples t-test indicated that the spatial ability scores were significantly higher for the preservice secondary teachers than for the preservice elementary teachers, t (63) = -3.186, $p < .01$, Hedges' g = .80. Results also indicated that ACT scores for preservice secondary teachers were higher than ACT scores for preservice elementary teachers t (63) = 3.178, $p < .01$, Hedges' g = .82 (see Table 3).

Table 3

T-test for Spatial Ability and ACT of PSTs

Construct	Secondary (*n*=26) *M (SD)*	Elementary (*n*=39) *M (SD)*	*t*	*df*
Spatial Ability	13.88 (3.30)	10.95 (3.85)	-3.19	63
ACT	25.54 (3.40)	22.59 (3.83)	-3.18	63

Discussion

In this study, we examined PSTs' mathematics self-efficacy and spatial ability to determine the strength of association between these constructs and if there were differences between levels reported by elementary and secondary PSTs. The descriptive statistics reported for the first research question found that both elementary and secondary PSTs scored relatively low on the spatial reasoning test, with an overall mean of 12.12 out of 20. However, these results are consistent with previous research findings with undergraduate students (see Bodner & Guay, 1997). Additionally, we found that the overall mean for PSTs' mathematics self-efficacy was high (5.84 out of 9). This finding is also consistent with prior research indicating that PSTs tend to self-report positive beliefs such as mathematics teaching self-efficacy (Briley, 2012; Giles et al., 2016).

With regard to the second research question, results indicate that mathematics self-efficacy and spatial ability are positively associated for PSTs. This result might not be surprising given that achievement and other skill-based assessments have been shown to correlate or predict mathematics related beliefs (Giles et al., 2016; Paunonen & Hong, 2010; Uttal & Cohen, 2012). For example, Paunonen and Hong (2010) found that self-efficacy was predictive of performance on spatial tasks. However, it is curious that a stronger correlation was not found between the two constructs given the connection between self-efficacy and performance in prior research.

The third question explored the differences in spatial ability and mathematics self-efficacy between elementary and secondary PST. We found that secondary PST's reported significantly higher levels of self-efficacy, Mathematics Tasks and Mathematics Courses, than elementary PSTs. This finding may be due to varied experiences between the groups of PSTs. For example, secondary PSTs take a large number of mathematics courses, which provides them with more opportunities that could build spatial reasoning. The finding that elementary PSTs' MSE overall score was lower than secondary PSTs might indicate a connection between certain students' choice to pursue elementary education instead of secondary education, given prior work indicating students' mathematics self-efficacy was a strong predictive effect over choices in college (Parker et al., 2014).

Limitations and Implications

One limitation of this study was that elementary and secondary PSTs were not necessarily at the same point in their education program. Some PSTs were earlier in their program than others; although, all PSTs were given the survey and assessment in a mathematics methods course. Another limitation is that the sample population of this study was predominately female. According to many previous studies (e.g., Maeda & Yoon, 2013) females have been noted to have lower spatial ability scores on timed mental rotation tests, which might have influenced the results found.

Gauging PSTs' mathematics self-efficacy and spatial ability throughout their training programs can give teacher preparation programs a means to evaluate the growth of these constructs over the course of their college coursework. For example, pre- and post-tests could illuminate the effectiveness of methods courses' intentional interventions targeting spatial ability. Methods courses including activities to support building spatial ability could also be used by PSTs in their future classrooms, which is important to consider given the connection between spatial ability and children's mathematics achievement (Gilligan et al., 2017). Uttal and Cohen (2012) also suggested that widespread implementation of spatial ability training could have an overall small positive effect on retention outcomes in STEM majors in comparison to the small cost of such training.

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MATHEMATICS-RELATED EXPERIENCES AND BELIEFS OF PRESERVICE ELEMENTARY TEACHERS

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Mathematics-related beliefs have been shown to have a significant impact on teachers' pedagogical decisions and instructional practice. There has been an increased focus describing the development and content of preservice teacher beliefs. This study reports the beliefs of elementary preservice teachers and summarizes learning experiences participants claim had a significant impact on their view of mathematics and mathematics learning. Participants in the study indicated they held beliefs that were consistent with a cognitive constructivist view of mathematics and largely described learning experiences which were consistent with those beliefs.

Introduction

Researchers (e.g., Minarni, et. al., 2018) have shown that teaching practice is highly influenced by a teacher's beliefs. Results such as these have led researchers to explore the mathematics-related beliefs of preservice elementary teachers (PETs) in order to promote beliefs that may lead to effective teaching practices. However, these studies (e.g., Purnomo et al., 2016) have focused on characterizing PETs' beliefs and left the development of those beliefs largely unexplored. Pajares (1992) argued that PET beliefs about teaching mathematics are likely linked to beliefs formed from learning experiences that occurred throughout their PK-12 education. Thus, the purpose of this research is to describe significant educational experiences that preservice elementary teachers say had an impact on their mathematics learning. Further, this research will describe the mathematics-related beliefs of PETs and begin to explore relationships between past experiences and current beliefs. Therefore, the specific research questions guiding this study were: What mathematics related experiences tend to stand out for preservice teachers? What beliefs do PETs hold about mathematics and the teaching and learning of mathematics?

Background Literature

Mathematics-Related Beliefs

According to Philipp (2007) beliefs are defined as "psychologically held understandings, premises, or propositions about the world that are thought to be true" (p. 259). Teachers' mathematics-related beliefs have been linked to their instructional decisions (e.g., Stipek et al., 2001; Wilkins, 2008), how they interact with mathematics curriculum (Collopy, 2003), and student achievement (e.g., Šapkova, 2014). In mathematics education research, mathematics-related beliefs are often grouped into two related, yet distinct sets of beliefs: beliefs about the nature of mathematics and beliefs about the teaching or learning of mathematics. Thompson (1992) characterized beliefs about the nature of mathematics as a "teacher's conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics" (p. 132). Grigutsch et al. (1998) suggested four characterizations of a teacher's beliefs about the nature of mathematics: formalism-related, scheme-related, process-related, and application-related orientation. In addition, they suggest that the formalism-related and scheme related orientations both suggest a *static* view of mathematics where the process-related and application-related orientations correspond to a *dynamic* view of mathematics.

According to Staub and Stern (2002), beliefs about teaching and learning mathematics fall into two categories: a direct-transmission view and a cognitive constructivist view. A direct-transmission view of teaching and learning suggests that students will learn as long as the teacher provides opportunities for adequate practice and well-structured learning environments. Typically, a teacher that holds a direct-transmission view of teaching does not make a distinction between understanding and procedural fluency. One who holds beliefs consistent with a cognitive constructivist view; however, emphasizes learning experiences based on the needs of the learner and believes "understanding is based on the restructuring of one's own prior knowledge from the very beginning of the learning process" (Staub & Stern, 2002, p. 345).

Although research has been done which attempts to characterize beliefs of various groups (e.g., preservice teachers), there has been little research that explores what led to the development of the mathematics-related beliefs that preservice teachers hold. In his foundational work characterizing the nature of beliefs, Abelson (1979) suggested that beliefs are often grounded in episodic material and personal experience. Furthermore, according to Pajares (1992), early experiences have the potential to impact belief formation because beliefs have the potential to be self-fulfilling. Because beliefs likely have an impact on behavior, they are able to influence our experience, which in turn can reinforce the original belief. This characteristic of beliefs is particularly important for beliefs related to teaching. In most cases, students begin college with relatively few experiences related to their chosen field. As a result, much of their beliefs can be formed as they learn about and gain personal experience in their chosen fields.

This, however, is not the case for PETs. As Nespor (1987) suggests, one's significant experience as a learner or a particularly influential teacher may lay at the root of one's views of teaching.

Mental Model Theory

This research is grounded in both mental model theory and the use of drawings as a way to characterize PETs' experiences with mathematics. In commenting on mental model theory, Johnson-Laird (1983) explains that mental models are cognitive structures and that an individual constructs a representation of a phenomena in their world based on how they perceive, imagine, and recall that phenomena in order to make sense of their world. Thus, an individual constructs mental models about mathematics through their experiences with mathematics and then uses these mental models to interpret, understand, and reason about their world (Jacob & Shaw, 1998). It follows that the mental models that one constructs about mathematics will influence one's expectations, experiences, and how they acquire new knowledge.

Researchers have indicated, "a person's mental model reflects his/her belief system, acquired through observation, instruction, and cultural influences" (Libarkin et al., 2003, p. 123). Additionally, Nespor (1987) differentiated between beliefs and knowledge indicating that an individual will meaningfully store knowledge but their beliefs are drawn from their lived experiences such as one's experiences with mathematics. This allows one to explore PETs' beliefs through their mental models. Participant-created drawings that capture an individual's mental images of their experiences have been used to explore their mental models. For example, drawings have been used to analyze PETs' mental models of themselves as a teacher of science (Thomas et al., 2001) and as a teacher of mathematics (Utley & Showalter, 2007) as well as their representations of doing mathematics (Wescoatt, 2016), the environment (Moseley et al., 2010), and the work of an engineer (Hammack et al., 2020).

Methods

Participants

The participants in this study consisted of 22 female PETs enrolled in their second of two elementary mathematics methods courses in a large mid-western university. Nearly all (n = 21) of the participants identified as white, with the remaining student identifying as Black or African American. Participants ranged in age from 21 to 27 years with an average age of 21.5 years.

Instruments

***Three Experiences* Drawings.** In order to capture learning experiences that may have influenced the participants' mathematics-related beliefs, each participant was asked to complete a *Three Experiences* protocol which is composed of two steps. First, participants list three experiences that they feel had a significant impact on them as a learner of mathematics. Second, participants illustrate and reflect upon these experiences in order to describe feelings and emotions associated with those experiences.

TEDS-M. To measure mathematics-related beliefs, this study utilized the TEDS-M Beliefs about Mathematics and Mathematics Learning instrument, which is composed of five subscales, grouped into three categories. First, beliefs about the nature of mathematics are measured using the Mathematics as a Set of Rules and Procedures subscale (six questions) and the Mathematics as a Process of Enquiry subscale (six questions). Second, beliefs about learning mathematics are measured using the Learning Mathematics through Following Teacher Direction subscale (eight questions) and the Learning Mathematics through Active Involvement subscale (six questions). Finally, the beliefs about mathematics achievement are measured by the Mathematics as a Fixed Ability subscale (eight questions). Together these scales consist of 34 Likert-type questions where participants respond to questions by indicating their level of agreement with the statement using a 6-point Likert scale (1: strongly disagree to 6: strongly agree). The scales were designed in such a way that a response of a five or six was considered an endorsement of the statement (Tatto et al., 2012).

Analysis and Results

Research Question 1

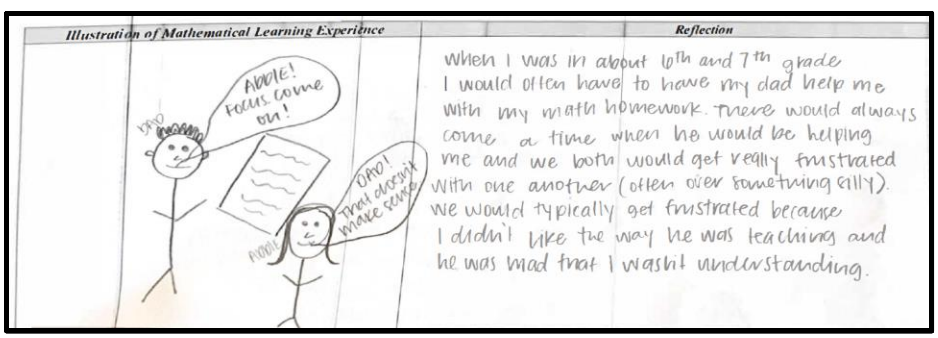
To analyze drawings, the research team first independently used open coding to identify and label initial codes to units of data within each student's drawing and description. Next, the two independent researchers' codes were discussed until a consensus was reached for each code. Using axial coding, researchers identified relationships among the open codes to group open codes into categories. Since many of the experiences also described strong feelings and emotions, each experience was coded holistically as either positive, negative, or neutral.

Overall, PETs' experiences with mathematics tended to be either positive (38.3%) or negative (40.0%); however, nearly one-fourth (21.7%) were either truly neutral or had a balance of negative and positive aspects within the depiction of the experience. Additionally, examination of the grade levels associated with the three experiences that came to mind revealed that one-third of all experiences reported were at the elementary level and one-fourth at the high school level. In another one-fourth of the responses, the grade level was not able to be determined and the experience was presented as occurring repeatedly. For example, one student commented that she noticed that she would solve problems "different than others around me" which lead to "the feeling of not doing problems right". However, this description was not tied to a particular experience but instead reflected her feelings throughout her K-12 experiences.

Several major categories arose from the coding process including parents, teachers, time, attitudes, and performance perceptions. Responses that fell within the parents' category tended to indicate that their parents were helpful and supportive, such as "my dad used to hang up posters around the house to help me learn my math facts...positive math memories." Some indicated that parents were not helpful because they were not aware of how the teacher was teaching or that parents helping with homework often led to an emotional response such as "some yelling and some crying." Figure 1 provides an example of a response that fell within the parent category as not helpful. The image shows, a dad attempting to support their child with their mathematics homework and the frustrations that often arose.

Figure 1

Sample Student Illustration and Reflection upon a Mathematical Learning Experience



The category of teachers fell into two major subcategories related to either teacher instruction or the learning environment. Students who reported positive experiences related to teacher instruction described liking step by step modeling of examples or getting to use manipulatives to help them understand. However, negative experiences were more prevalent. Responses revealed that students had negative experiences with timed activities (particularly the very common timed basic facts tests), the teacher "being bad" at teaching mathematics, pacing by the teacher being too fast, and several students indicated that the teacher expected them to solve problems using a prescribed process. Similarly, some students described positive or negative experiences related to the learning environment established by the teacher. For example, some students indicated that they recalled teachers who were kind and supportive, had positive attitudes towards students and their learning, and made them feel they were valued as a student. However, other students described learning environments that made them feel stupid, frustrated, and uncomfortable asking questions.

Time was another theme that emerged. Time emerged primarily in relation to timed basic fact tests. Overwhelmingly, students stated that these timed tests generated anxiety and frustration about mathematics. However, some students did reflect positively on timed activities because they enjoyed the competition that inevitably arose. Attitudes were also prevalent throughout the responses. Students expressed attitudes related to a lack of confidence (e.g., "not confident in my ability), feelings of anxiety, and their view of the usefulness of mathematics in their life (e.g., learning contexts such as money and time). Lastly, some students' responses were associated with perceptions of their performance in learning mathematics. For example, students' comments included "I failed to understand a single concept of what was happening," "nothing made sense," or "oftentimes when I am doing math I feel that I am behind the rest of the class."

Research Question 2

To explore PETs' beliefs about mathematics and learning mathematics, TEDS-M beliefs data was imported into a statistics software, coded, and then descriptive statistics were calculated (see Table 1). Due to the small sample size, the Cronbach Alpha values were acceptable but slightly smaller than the established instrument. Results indicate that PETs endorse the belief that mathematics is a process of enquiry and that mathematics learning occurs through active involvement. Further, the PETs disagreed with the belief that mathematics learning should be teacher-directed and that mathematics is a fixed ability. Finally, PETs tended to slightly agree with the belief that mathematics is a set of rules and procedures.

Table 1

Mathematics-Related Beliefs (n = 22)

Construct	*M*	*SD*	*Min*	*Max*	*Study α*	*Established* α*
Rules and procedures	4.31	0.61	3.00	5.00	0.74	0.94
Process of enquiry	5.33	0.47	4.67	6.00	0.78	0.91
Teacher direction	2.47	0.82	1.25	4.13	0.84	0.86
Active involvement	5.07	0.52	4.17	6.00	0.71	0.92
Fixed ability	2.18	0.93	1.00	4.25	0.84	0.88

*Tatto, (2013)

Conclusions

First, results of this study indicate that PETs believe mathematics to be a process of enquiry that is learned through active involvement. This suggests they hold beliefs that are consistent with a cognitive constructivist view of mathematics learning (Staub & Stern, 2002). However, there is evidence that PETs also hold some beliefs that are inconsistent with this view as they tended to agree with the belief that mathematics is a set of rules and procedures. Second, PETs indicated both positive and negative past experiences had significant impacts on their learning of mathematics. Experiences indicated by PETs tended to center around parents, teachers, time, attitudes, and performance perceptions. In addition, analysis of these experiences suggests that descriptions of learning experiences consistent with a cognitive constructivist approach were largely associated with positive feelings and attitudes, while those that were not consistent with a cognitive constructivist approach were often associated with negative feelings and attitudes. Future research will include a larger sample size and continue to explore potential relationships that may exist between PETs' beliefs and their prior learning experiences. Additionally, given the number of negative experiences that participants described, future research could explore why these PETs are pursuing a degree in elementary education.

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SUPPORTING CONNECTIONS TO TEACHING IN AN UNDERGRADUATE CALCULUS COURSE

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The Mathematical Education of Teachers II report by the Conference Board of the Mathematical Sciences (2012) recommends that undergraduate programs enhance prospective secondary mathematics teachers' (PSMTs) understanding of connections between the advanced undergraduate mathematics content and the mathematics they will teach. This paper examines the connections to teaching made by one instructor and one undergraduate PSMT after implementation of two calculus lessons aimed at supporting connections to teaching. Each lesson embedded approximations of practice tasks in the learning of calculus content. Findings suggest that these lessons enabled both deepening mathematical content knowledge and insight into the work of teaching.

Calculus is a gateway course for science, technology, engineering, and mathematics majors and is part of every PSMTs' program of study. It is quite common to embed applications to physics, biology, or chemistry in calculus courses, but despite recommendations from the Conference Board of the Mathematical Sciences' (CBMS) Mathematical Education of Teachers II (MET II) report (CBMS, 2012), applications to teaching that make explicit connections between advanced content in mathematics to mathematics taught in secondary schools are much less common (e.g., Lai & Patterson, 2017). As the MET II report encourages making connections to teaching throughout a prospective teacher's mathematics program of study, the *Mathematical Education of Teachers as an Application of Undergraduate Mathematics* (META Math) project's initial focus has been on developing lessons in calculus, discrete mathematics, abstract algebra, and statistics, typical undergraduate mathematics courses in which PSMTs enroll.

For each course, META Math developed two lessons focused on building connections to teaching via applications of mathematics to teaching and collected data on implementation at over 14 universities nationwide.

As part of a larger study, this paper explores one instructor's experiences implementing two META Math lessons in calculus and one PSMT's experiences with the lessons. We focus on how the lessons support instructors' and undergraduates' awareness of connections to teaching. We explore the following research question: In what ways does the use of applications to teaching in calculus support an instructors' and an undergraduates' awareness of the role of connections to teaching in a mainstream mathematics course?

Background and Theoretical Perspective

Mathematical Knowledge for Teaching (MKT) was studied by Ball et al. (2008) as being "the mathematical knowledge needed to carry out the work of teaching mathematics" (p. 395). While the MET II (CBMS, 2012) recommends that future teachers make connections between advanced and school mathematics throughout their mathematics program, there are few studies that show that future teachers are making these connections in their undergraduate mathematics programs. Wasserman (2018) found that "teachers and their students appear to gain little from a teacher's study of advanced mathematics" (p. 4). Other researchers have noted that PSMTs have found the advanced content to be disconnected to what they will one day teach, or even that they do not understand the foundational concepts at a deep enough level to teach them (e.g., Goulding et al., 2003; Wasserman et al., 2018).

For university mathematics instructors, identifying and using appropriate resources poses unique challenges as many are unfamiliar with MKT, how to highlight connections to teaching in a mainstream course, how to identify applications to teaching in their extant curriculum, or the role of applications to teaching for PSMTs (Álvarez & Burroughs, 2018; Lai, 2016).

Ball et al. (2008) proposed that a skill necessary for teachers was to be able to "hear and interpret students' emerging and incomplete thinking" (p. 401). Incorporating experiences throughout teacher preparation programs that engage undergraduates in practices used in mathematics teaching can take the form of including approximations of practice tasks in mainstream courses. Grossman et al. (2009) describes these tasks as "opportunities to engage in practices that are more or less proximal to the practices of a profession" (p. 2058; see also Álvarez et al. 2020). For example, Ghousseini and Herbst (2016) use constructed dialogues and Campbell et al. (2020) employ "planted errors" to approximate the work of teaching mathematics.

Methodology

To incorporate mathematics teaching connections as a legitimate application area of undergraduate mathematics, the META Math project developed inquiry-based lessons for calculus that address MKT and the recommendations of the MET II (CBMS, 2012) report. The lessons consist of content-specific and pedagogical connections to engage undergraduates in connections to teaching. Several tasks in the lessons require students to analyze a hypothetical student's work or choose or pose guiding questions for probing student thinking (e.g., Figure 1).

Figure 1

Assessment Example from Newton's Method

2. A student tried to use Newton's method to find the zeros of the function $f(x) = x^3 - 3x + 1$ by beginning with an initial guess of 1. She was unsuccessful and asked you for help.
   (a) Why was she unsuccessful? Explain, using tangent lines, what is causing this behavior.
   (b) Write two questions you can ask the student to help her correct her work. Explain how your questions might help the student.

The calculus lessons used by the instructor and PSMT in this study focused on inverse functions and Newton's method. *Inverse Functions* reviews commonly taught methods of finding an inverse function and explores how these methods influence the formulation of derivatives of inverse functions. *Newton's Method* introduces undergraduates to an iterative method of approximating the zeros of a function by looking at hypothetical students' work of using tangent lines at points on the function to see where the line intersects the x-axis. Both of these lessons incorporate opportunities to analyze another student's work.

The META Math lessons incorporate five types of connections to teaching between college-level mathematics and knowledge for teaching school mathematics (see Table 1). Arnold et al. derived these connections from Ball et al.'s (2008) six categories of MKT (as cited in Álvarez et al., 2020). Both lessons consist of an activity-based lesson (separated into pre- and class-activities), homework questions, and assessment items. Instructors receive a detailed annotated lesson plan (ALP), which serves as a guide for implementing the lessons effectively.

Table 1

Five Types of Connections to Teaching (Álvarez et al., 2020)

Connection	Description
Content Knowledge (CK)	Undergraduates use course content in applications or to answer mathematical questions in the course.
Explaining Mathematical Content (EC)	Undergraduates justify mathematical procedures or theorems and use of related mathematical concepts.
Looking Back / Looking Forward (FB)	Undergraduates explain how mathematics topics are related over a span of K-12 curriculum through undergraduate mathematics.
School Student Thinking (ST)	Undergraduates evaluate the mathematics underlying a student's work and explain what that student may understand.
Guiding School Students' Understanding (GSU)	Undergraduates pose or evaluate guiding questions to help a hypothetical student understand a mathematical concept and explain how the questions may guide the student's learning.

Setting and Participants

In the Fall 2019 semester two instructors at two different universities implemented both first-semester calculus lessons in their calculus courses. The universities are both large public research universities in the Midwest and Southeast United States, respectively. As part of their participation in the project, instructors were also invited to three interviews that occurred during the semester. All undergraduates in these courses, as part of their regular coursework, completed all parts of each lesson. We invited a subset of consenting undergraduates to participate in an hour-long semi-structured interview at the end of the semester. From a total of 63 consenting undergraduates, six undergraduates from each site consented to participate in interviews. For this paper we will focus on one instructor, Bruce, and one of his undergraduate students, Kayla.

Bruce has taught calculus for the last 15 years. At his institution, calculus is not a coordinated course. His department has 174 mathematics majors, about 20% are PSMTs. Kayla intends to teach middle school mathematics. She is an interdisciplinary liberal studies major with a minor in mathematics. We chose Bruce as a representative case as he is an experienced calculus instructor with no prior experience using these pedagogical ideas. Kayla was chosen as a critical case as the only self-identifying PSMT in the course. Participants were given pseudonyms.

Data Collection and Analysis

Both instructor and undergraduate interviews lasted between 45-60 minutes and were audio-recorded and transcribed. During the undergraduate interviews, students re-examined their work on the assessment items from each interview (e.g., Figure 1). While reconsidering their work, they provided explanations of their thought processes where appropriate, considered alternative approaches, and discussed the potential connections to previous math content. Interview questions were often posed through the lens of connections for teachers, but interviewees discussed their own perceptions of the assessment items and connections to teaching emphasized in the lesson regardless of their intent to formally teach in a classroom environment.

There were three instructor interviews, two occurred shortly after each lesson was taught, which focused on how the implementation went and how prepared instructors felt to teach the lesson based on the provided resources. At the end of the semester, the third interview reviewed both lessons, eliciting instructors' views of the project and the five types of connections.

We used thematic analysis (e.g., Braun & Clarke, 2006; Nowell et al., 2017) to qualitatively analyze the interview transcriptions. Each interview was first coded for the five connections to teaching. These codes were then expanded inductively with any emergent thematic ideas.

These additional codes tended to relate to teaching, implementation of the lesson, or the format of the activities. Once each lesson was coded independently, we compared codes until we were in agreement. Less pervasive codes, such as those that did not relate to the types of connections or were only present in one interview, were eliminated or integrated into broader categories.

Results

Bruce

While preparing for both lessons, Bruce reported carefully reading through the ALP a week before implementation. Bruce had students complete the pre-activity in class to gauge, based upon their work, the undergraduates' prior knowledge of the topics. During the next class, he began with a brief discussion of the pre-activity before undergraduates worked on the class activity in groups. Bruce's class took two class periods to complete the activity. He found that the materials, especially the ALP, prepared him to implement the lesson, although he was unprepared for how little his students remembered inverse functions from past math courses.

While discussing his views on the five types of connections and how they influenced his teaching, Bruce points out that *ST* and *GSU* were connections he had not previously considered using in his mathematics teaching. He said that he does not often consider questions like that in Figure 1 because they are not "really relevant to [him] directly." After seeing how his undergraduates had responded to these questions, he added that if he had even one future teacher in his class he would "be thinking more clearly about [those connections] and trying to make ties [to school mathematics]." When discussing his experiences implementing *Inverse Functions*, he explained that he often hesitates to show others' work because of time constraints and privacy issues, but by looking at hypothetical students' work, he found it "valuable and worth [his] time" to see students interacting with others' work and that "it can be really useful for students to observe other students work. As opposed to just my pre-scripted, professor style writing."

Bruce also discussed how he sees a difference in the way he might present content with these connections in mind, especially *CK* and *FB*. He mentions that normally, as a mathematics professor, he wants "to emphasize [the] fancy ways of solving hard things" but that is not always "what's most useful for the students." He then explains that for future high school teachers having a strong background in these connections to then be able to teach the content to school students would be most helpful. He learned better scaffolding techniques for tasks when implementing the lessons and realized the importance of scaffolding material, especially for future teachers. Building connections to teaching resonates for Bruce as it is coupled with the rigor of an undergraduate course. For *Inverse Functions,* he acknowledges the appropriateness of the content for all of his undergraduate students, specifically that this lesson is "crucial for anyone whose major requires [calculus]." As mentioned, he also realized "how little about inverse functions [his students] remember." This allowed him to provide support from which to build the class activities. For *Newton's Method* he said that he was "going to entirely use the *Newton's Method* worksheet… [it] was a great take-away, content-wise." Overall, Bruce recognized that these lessons were both appropriate for future teachers and other students.

Most of Bruce's undergraduates were not PSMTs, but he indicated that he wanted to motivate these connections for all of his students by emphasizing that many fields require explaining content or analyzing someone's thinking; so, for non-PSMTs, Bruce highlighted communication skills as important outside of a classroom environment. He found the lessons easily adaptable and suitable for all students while maintaining fidelity to the connections to teaching.

Kayla

Kayla plans to teach middle school mathematics after getting her master's degree in education. We focus on Kayla since she was the only PSMT in Bruce's class. Kayla participated in both class activities for the lessons and then consented to participate in an end of the semester interview about her experiences during those days where the lessons were implemented.

Kayla said that she "loved" having the experience with student work and posing guiding questions. She saw a direct connection to teaching since she expects to see children's work that displays unfamiliar strategies or methods or that may not be thorough enough to readily determine the strategies or methods used. She also indicated that she had fun looking at student work and decoding "what they have done in their thought processes" when it was not possible to speak with them directly. Moreover, Kayla saw the depth of the mathematical thinking required in the lessons as an important feature for her as a prospective teacher since "there's always going to be the kid who's curious in your class as to why things work, and I think it's really important to … have some understanding and then be able to be like well I, maybe I don't know why, but why don't we figure it out together." Kayla also commented, "I think a lot of people who maybe don't have upper level math understanding will shut that down out of fear of like, not knowing the answers themselves." This was linked to stunting school students' interest in math and contrasted with “being able to like explore stuff like this and be comfortable with exploring it … really, it's gonna benefit students in the future, being able to help them explore math too.”

Kayla discussed that explaining mathematical content in the lessons provided insight into connections to teaching. When engaging in the *Inverse Functions* activity she recognized the value in explaining her work as it mirrors what she would want her future students to do. Looking forward she adds that, “as a future teacher you want your kids to know the meaning of the work that’s behind what they’re doing, you don’t just want to memorize steps.” She then explains that this lesson reinforced ideas that she would want her future students to have, such as a good grasp of definition usage and conceptual understanding versus overreliance on formulas.

Kayla found the structure of both lessons to be good models for an inquiry-oriented class environment. She commented that she could apply the scaffolded approach in her future lessons. The chance for students to “explore the meaning [of these topics] for themselves” tied into her belief that the concepts will “stick with kids more” when actively engaged in learning. Having students actively work through the steps aligned with her belief in having students “understand something for themselves and make connections.” Recognizing the connections to high school, Kayla explains that understanding “basic algebra and … graphing and lines” would prepare a student for investigating Newton’s method. She expressed that revisiting and applying these connections would better prepare her for explaining elementary ideas to her future students.

Discussion and Conclusion

Both Bruce and Kayla reported that concepts in the lesson helped them see connections to teaching, especially the problems that presented a hypothetical student’s thinking which then had to be analyzed. Kayla expresses not only her appreciation for the student thinking questions addressing the needs of future teachers’ development of communication skills, but also addresses the connection *Looking Back/ Looking Forward* discussing how knowing advanced content can help a teacher explain concepts to curious students that may require more justification.

Kayla found that the lesson structure helped her see the need to scaffold concepts to students, and Bruce said that the use of scaffolding benefited all students, especially PSMTs. Each reported that the approximation of practice tasks supported their awareness of connections to teaching. Bruce recognized the importance of including these tasks in his course and how they address the needs of all students. Using the materials raised his awareness of the needs of PSMTs in his courses and the importance of integrating these ideas throughout the curriculum.

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Reestablishing Connections for Preservice Teacher Field Experience Support

MAKING MATHEMATICAL CONNECTIONS BY BUILDING COMMON GROUND

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Successful everyday activity with others is possible by building shared understanding (i.e., common ground). During a mathematical lesson, building common ground is a challenging but essential factor that affects the mathematics learning of a classroom community. In this paper, I analyzed the micro-interventions a secondary mathematics student teacher made to build common ground with students to support them in making a mathematical connection. The analysis supports the assertion that teachers can begin to specialize their everyday practice of building common ground for teaching given time and experience.

Introduction and Relevant Literature

Speakers will adapt their speech and gestures to be understood by addressees, which some refer to as *audience design* (Clark & Carlson, 1982). The speaker's adaptation is an attempt to build or maintain *common ground*, or a shared mutual understanding. Common ground is essential for all joint activity (Clark, 1996), such as whole-class mathematical inquiry (Staples, 2007). Much of the research on common ground has taken place in clinical research settings, and so, I briefly reference key findings from relevant clinical studies for interpreting the data. For instance, speakers will use commonality assessments to monitor and adjust for the needs of addressees (Horton, 2005). In conversation, for example, a speaker might ask, "Remember what happened last time?" The speaker's question assesses what they share in recalling a previous event. Speakers will also use more gestures when they know that the information is highly relevant to addressees (Kelly et al., 2011). Speakers may furthermore adjust their gestures (e.g., change the speed or form) when facilitating addressees' identification of specific visual features (Peeters et al., 2015).

There is a growing number of research studies investigating common ground in mathematics classrooms. Staples (2007) described several strategies that an expert secondary mathematics teacher used to establish and monitor common ground during whole-class collaborative inquiry (e.g., pursuing discrepancies), which she organized into three themes: (a) creating a shared context, (b) maintaining continuity over time, and (c) coordinating the collective. Alibali and colleagues focused their research on how teachers' gestures establish and maintain common ground during instruction (e.g., Alibali et al., 2013, 2019; Nathan et al., 2017). For instance, to build common ground, Alibali et al. (2013) found teachers will increase their use of gestures in response to students displaying or expressing lack of understanding.

These studies on common ground in mathematics classrooms describe how *experienced* teachers build and maintain common ground. They, however, do not provide insight into whether *novice* teachers can begin to build or maintain common ground, if at all. Therefore, I examined the practice of secondary mathematics student teachers to see if they would make similar attempts to building common ground. I focused on classroom interactions of making mathematical connections because explicit attention to connections are generative of students' learning (Hiebert & Carpenter, 1992).

Theoretical Framework

Clark (1996) distinguished different domains for common ground: communal, personal, and incremental. For this paper, personal and incremental common ground are particularly relevant. Personal common ground is the shared knowledge between speaker and addressee(s) resulting from prior experience or current situation. For example, a teacher and students may share a common understanding of the meaning of a fraction developed from previous lessons. Incremental common ground is the moment-by-moment shared understanding that the speaker and addressee(s) establish in conversation. For example, a teacher and students may gradually build a shared understanding of equivalent fractions over lessons.

According to Horton and Gerrig (2002), the details of speakers' experiences need to be accounted for when making claims about the presence of audience design because "Speakers may intend quite sincerely to tailor their productions for a specific audience, but lack the knowledge or resources to carry out these intentions fully" (p. 605). Horton and Keysar (1996) also argued a speaker's initial utterances are not always in consideration of common ground, but rather the speaker will monitor and adjust for the addressees' needs.

Methodology

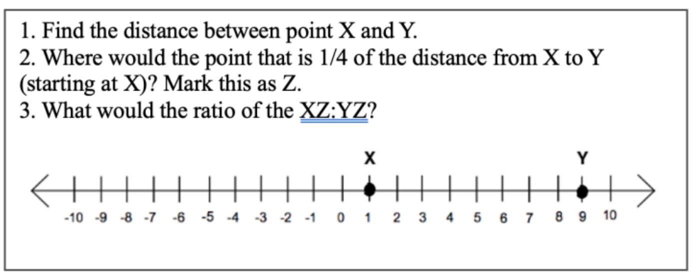
Context and Data

The study was an instrumental case study (Stake, 2003). Melissa and Robin (pseudonyms) were selected for this study from their participation in a larger research project that followed a cohort of secondary mathematics teachers in their teacher education program. During their student teaching, Melissa and Robin co-planned and co-taught two sections of an advanced ninth grade mathematics course. Data included lesson materials and video-recordings across the same unit of instruction for each section. In this paper, I present the initial findings from the second lesson from the unit. Robin was the lead teacher for the lesson for both classes.

The goal of the lesson was for students to be able to find a point on a directed line segment that partitioned the line segment in a given ratio. For example, if points $A\ (-5, -6)$ and $B\ (4, 11)$ formed a line segment, then find a point on the line segment from A to B that partitioned the line segment in a ratio of 2 to 3. Students were to explore partitioning a line segment on a number line and then generalize a formula to partition any line segment on a number line in a given ratio. Figure 1 is a recreation of the exploratory task. After generalizing a formula for the number line, students were to derive a formula to partition any line segment in a coordinate plane, which the teachers referred to as the partitioning formula. The formula resulted in a point, $\left(\frac{a}{a+b}(x_2 - x_1) + x_1, \frac{a}{a+b}(y_2 - y_1) + y_1\right)$, that partitions a line segment formed by points (x_1, y_1) and (x_2, y_2) into a ratio from a to b. After deriving the partitioning formula, the teachers assigned students problems to practice applying the formula (work period).

Figure 1

Exploratory Task for Partitioning a Number Line.



Data Analysis

I began the analysis by transcribing all the video-recordings for each lesson. In phase one, I iteratively watched the video-recording of each class in 5- to 10-minute increments and noted the content-related episodes in the lessons. A content-related episode included activities such as discussing a solution to a mathematical task and not the day-to-day operation of school (e.g., checking attendance). In phase two, I created side-by-side transcripts of similar content-related episodes across the two class sections. After re-watching each episode, I enriched the transcripts by describing teachers' gestures and noting representations teachers or students displayed. I also noted trouble spots, which included students' questions seeking clarity of a teacher's meaning, students' incorrect responses to a teacher's question, and uncertainty or hesitancy in students' responses. In phase three, I created memos next to the side-by-side episode transcripts to describe the contexts, conditions, and teacher actions using the constant comparative method (Glaser & Strauss, 1967) for each episode. In phase four, I compared across memos noting similarities and differences in teachers' responses to build and establish common ground during the episodes and how they aligned, if at all, to results from clinical and field research.

Results

Several trouble spots (e.g., students expressing uncertainty) occurred during the lesson. In this paper, I focus on one trouble spot that arose in the first enactment of the lesson but not in the second. I selected this trouble spot as it related to a mathematical connection Melissa and Robin wanted students to make during the lesson: the ratio 1 to 3 is related to the fraction one-fourth, as similarly described in the "overlapping" model (Clark et al., 2003).

The Trouble Spot: The Relationship between $\mathbf{1:3}$ and $\mathbf{1/4}$

While working with students on generalizing a formula to partition a line segment on the number line, Robin shifted students' focus to the ratio the students found in the exploratory task, in what appears to be the teacher wanting students to make a connection between the ratio 1 to 3 and fraction one-fourth. Table 1 provides transcripts of the discussion. In this episode, Robin made the connection that the ratio 1 to 3 is related to the fraction one-fourth. There is, however, no explicit reason given for the connection. Later in the lesson, Robin explicitly stated a general relationship that the ratio a to b is related to $\frac{a}{a+b}$, but again, does not give any reasoning for why the two mathematical objects are similar. While students did not explicitly state any uncertainty during this moment in the lesson, there were several moments of silence (Lines 1.3, 1.12, and 1.15). Later episodes in the lesson provide further evidence that some students were uncertain about the reason for the connection. For instance, Melissa later worked independently with two students on understanding the relationship between the ratio 1 to 3 and the fraction one-fourth during the work period. Also, when going over a problem with a ratio of 2 to 3, some students expressed confusion as to why the ratio is related to the fraction two-fifths.

Table 1

Transcripts of the First Enactment of the Lesson

Line	Speech transcript	Gesture transcript
1.1	Robin: Really quick, remember our ratio 1 to 3? That's what we got to.	Underlines ratio *1:3* on the board
1.2	Robin: Can we find, or can we think about a relationship between the ratio 1 to 3	Writes *1:3* on the board
1.3	Robin: and the fraction one-fourth? (4-second pause)	Writes *1/4* on the board

1.4	Robin: Like is there another way we could write that where it could kind of relate to this ratio?	Open palm held underneath 1:3
1.5	Student 1: Can you... can you do one-third or you can do one-fourth or one dot dot four?	
1.6	Robin: Well I'm talking about - so these are two different things, right?	Open palm held underneath 1:3 and 1/4 moving back-and-forth
1.7	Robin: We have 1 to 3 and then like one four, one-fourth, right?	
1.8	Student 1: You can do percentages, can't you?	
1.9	Robin: All I'm asking is that I want this to still stay as one-fourth	Open palm held underneath 1/4
1.10	Robin: but I want to see if there's any ways we can write it	Open palm held underneath 1:3
1.11	Robin: to where we have like a 1 and a 3	Both hands are open palms moving back-and-forth
1.12	Robin: in this. (1-second pause)	Open palm held underneath 1/4
1.13	Robin: Like using plus, minus, multiplication?	Both hands are open palms moving back-and-forth
1.14	Robin: So, we have a 1 on top,	Writes *1/* on the board
1.15	Robin: what can I write on bottom using these numbers that can give you 4? (3-second pause)	Open palm held underneath 1:3 and then sliding across to underneath 1/4
1.16	Robin: Are you all confused by that?	
1.17	Multiple students: [crosstalk: Several students can be heard expressing confusion]	
1.18	Robin: Wait. Student 2 what'd you say?	
1.19	Student 2: 1 plus 3.	
1.20	Robin: Okay.	Writes *1+3* on the board
1.21	Robin: Is that correct?	
1.22	Student 1: So, you're asking what can give you a 4?	
1.23	Robin: Yes. Okay, we're not going to go into this quite yet, but do you all see the relationship between this?	Open palm shifting from 1/4 = 1/(1+3) to 1: 3
1.24	Robin: Okay. I know it's kind of a confusing question, but we're going to need to know this in a minute.	

In the second enactment, Robin sought to create a shared context (Staples, 2007). I present the second enactment in Table 2 and then compare the two enactments. In this enactment, the connection is the same as the first enactment: the ratio 1 to 3 is related to the fraction one-fourth. There is, however, an explicit reason given by a student for the connection: "because, like one-fourth, there's like 4 parts to it." The student recognized that the fraction one-fourth and the ratio 1 to 3 were each a composition of four equal partitions.

Table 2

Transcripts of the Second Enactment of the Lesson

Line	Speech transcript	Gesture transcript
2.1	Robin: Okay and before we keep going from that, let's look at, let's look at our ratio that we found.	Draws a circle around 1:3 on the board
2.2	Robin: We found a ratio of 1 to 3. If we have a ratio that's 1 to 3, 1 part	Open palm on the left hand
2.3	Robin: to 3 parts, okay.	Open palm on the right hand

2.4	Robin: does that make sense? What, what's the total number of parts?	Both open palms held up facing each other
2.5	Multiple students: [crosstalk: Students can be heard giving answers of 3, 4, and 8.]	
2.6	Robin: Okay, what? So, I'm getting a lot of different answers.	
2.7	Student 3: 4.	
2.8	Robin: 4. Why 4?	
2.9	Student 3: Because	
2.10	Student 4: 1 plus 3	
2.11	Student 3: well, we started off like finding the distance, but then if you have 3 parts and you have like 1 part, then you add them together.	
2.12	Robin: Yeah okay, that's exactly right. So, um and then Student 4 you said…, what did you just say?	
2.13	Student 4: Oh, I just did 1 plus 3.	
2.14	Robin: Right, you're adding them. So how could I rewrite, so if I have the fraction one-fourth,	Writes *1/4* on the board
2.15	Robin: which is what we originally started with and I kind of want to write it in terms of the ratio that we found, how could I rewrite it to where it would make sense?	Writes = *1/* on the board
2.16	Student 5: 1 dot dot, wait no.	
2.17	Robin: In fraction form.	
2.18	Student 5: Oh um…	
2.19	Robin: Kind of incorporating this ratio.	Open palm held underneath 1:3
2.20	Student 5: One-third? I don't know.	
2.21	Robin: It has to, it needs to be equivalent to one fourth. You just -	
2.22	Student 5: Two-eights?	
2.23	Robin: Well, that is equivalent. You just said it, what do we do to get to the 4?	
2.24	Student 6: We added.	
2.25	Robin: You added.	
2.26	Student 5: Oh add.	
2.27	Robin: Right, so can we write this as like um… so can we write this as 1 over 1 plus 3.	Writes *1+3* on the board
2.28	Student 5: Yeah.	
2.29	Robin: Do you all see that? Okay. We're just trying to find a relationship kind of between these, okay.	
2.30	Student 5: Oh, I get it because, like one-fourth, there's like 4 parts to it.	
2.31	Robin: Yeah, exactly, exactly. So, we could like divide this into like 4 parts.	Open palm "chopping" the number line
2.32	Robin: Okay, there would be that ratio of 1 to 3, but it would be, so if I were to draw the parts like there, there, there, and there.	Writes vertical lines subdividing the number line into four equal parts
2.33	Robin: There's 1, 2, 3, 4 parts. Do you all see that?	
2.34	Student 7: Oh, I didn't know what parts she was talking about.	

Comparison of First and Second Enactment

In comparing the two enactments of the lesson, a small but seemingly significant difference occurred in how Robin began to build incremental common ground for the discussion of the connection. In the second enactment, Robin opens with a question that functioned as a commonality assessment (Horton, 2005). The question elicited several different answers from students and revealed that students lacked common ground for the total number of parts (Lines 2.4-2.6). Robin pursued the discrepancy in students' answers (Line 2.6) and asked a student to justify why she thought there were four total parts (Line 2.8). The pursuit of these discrepancies for the total number of parts would be productive for students to make a connection between the ratio 1 to 3 and the fraction one-fourth (Staples, 2007).

Similar to the first enactment of the lesson (Lines 1.2-1.20), Robin directed students to the connection that the ratio 1 to 3 is related to the fraction one-fourth in the second enactment (Lines 2.14-2.29). A student, however, did give a reason for the connection in the second enactment (Line 2.30). In response to the student's reasoning, Robin identified the four equal parts on the number line by drawing lines "chopping" the line segments into four equal parts (Lines 2.31-2.32). Robin's highlighting the four equal parts seemed to help focus some students' attention to the total number of parts (Line 2.34). Robin's gestures and highlighting are consistent with previous clinical research (e.g., Kelly et al., 2011; Peeters et al., 2015) and field research (Alibali et al., 2013, 2019). Robin changed the form of her gestures in the second enactment by pointing to the four parts to draw students' attention to the total parts, perhaps perceiving a need to communicate the four total parts to develop the intended connection.

Conclusion

While other studies have documented how experienced teachers establish and maintain common ground (e.g., Alibali et al., 2019; Staples, 2007), I provided an example of a micro-analysis of one student teacher's beginning practice to build and maintain common ground. The data and analysis provided evidence that beginning teachers, given opportunities to work with students on a task, can start to coordinate their actions to build and maintain common ground in similar tasks in the future. In this case, the student teacher used a commonality assessment and her gestures to establish common ground for the meaning of the total number of parts, which supported students in interpreting the intended connection during the second enactment of the lesson. However, it is uncertain if the student teacher consciously attended to common ground in the moment or perhaps discussed how to respond to the trouble spots later with her partner or mentor. To address this uncertainty, future investigations of this practice with teachers should consider including post-lesson interviews or video-stimulated recall interviews with teachers. Further micro-analysis may also provide insight into how this everyday practice of building and maintaining common ground becomes specialized for teaching.

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DEVELOPING PSTS' UNDERSTANDING OF THE MATHEMATICAL PRACTICES

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This paper explores how elementary preservice teachers (PSTs) understand the Standards for Mathematical Practice (SMPs) and identify them within elementary field placements. While observing mathematics lessons, the PSTs look for evidence of student engagement in and teacher facilitation of the practices. Understanding the SMPs and how to design and facilitate learning experiences that require students to engage in them is critical work for mathematics teachers. The patterns of learning revealed in the data from this exploratory mixed methods study can inform how mathematics teacher educators can support their PSTs' understanding of the SMPs as well as address common misconceptions.

While the Common Core Standards (NGA Center & CCSSO, 2010) were introduced ten years ago, many school systems are still working toward robust implementation of both the content and mathematical practice standards. When working with preservice teachers (PSTs), we believe it is essential to bring emphasis to the processes required of proficient mathematicians through a deep dive into the Standards for Mathematical Practice (SMPs). In considering those experiences, it is important that we examine the extent to which PSTs are able to identify when their students are engaging in the mathematical practices as well as identify when and how teachers facilitate the mathematical practices.

Purpose and Research Question

It is important that mathematics teacher educators (MTEs) consider strategic ways in which they prepare PSTs to understand the mathematical practices because it impacts their future K-12 students. Not only do future mathematics teachers need to have a strong grasp of content, they also need to know how to facilitate engagement of SMPs. Having a deep understanding of the SMPs is critical, so PSTs can plan and facilitate mathematics lessons that focus on the practices. By understanding how to use the SMPs to engage students in meaningful mathematics lessons, PSTs will be prepared to impact the development of K-12 students during field experiences and student teaching as well as when they are inservice teachers.

Given the importance of the role of MTEs developing deep understanding of the mathematical practices among our PSTs, we regularly reflect upon how our PSTs understand the SMPs as they identify them during their field experience. One assignment that we have our PSTs complete in their placement classrooms is to identify the SMPs they observe as their cooperating teacher conducts mathematics lessons. As we reviewed that assignment, we decided it would be helpful and informative to more formally examine the extent to which our PSTs are able to identify when *students* engage in and teachers *facilitate* opportunities for their students to engage in the mathematical practices. We focused our study by using the following research questions: (1) To what extent are PSTs able to identify the SMPs during a field experience? and (2) What misconceptions about the SMPs are prevalent among PSTs?

Relevant Literature and Framework

"The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students." (NGA Center & CCSSO, 2010, p. 6). Thus, the teacher plays a crucial role in strategically planning and facilitating learning experiences that will allow students opportunities to engage in particular mathematical practices. As part of teacher preparation, it is important to develop the ability of PSTs to understand the SMPs as well as to understand how their students engage in these practices and develop the skills to facilitate that engagement.

Students are learning about the concept of the SMPs in our university courses and then connecting it to experiences they have in the elementary classroom. Graybeal (2013) designed a similar study to ours, but PSTs used an iPad application to organize data. PSTs were encouraged to identify times in the lesson they were observing one of the eight mathematical practices. They spent one class period on each SMP and could record a small portion of the lesson, take pictures, or write a description to provide evidence of the SMPs. Our research differs from Graybeal's study because they only looked at how *students* were engaging in the practices, whereas we looked at how teachers *facilitated opportunities* for the elementary students to engage in the SMPs in addition to how students engaged in the tasks. Prior researchers (Graybeal, 2013; Johns, 2016; Wilkerson et al., 2018) have found teaching PSTs about the SMPs through real life examples, whether it was in field placements or using vignettes of student work and conversations in their methods class, helped PSTs better understand the SMPs. Our research differs from Wilkerson et al. (2018) because they used vignettes to help PSTs make sense of the SMPs whereas our students used video clips and observations in field placements. Johns (2016) implemented an activity where PSTs focused on constructing their own knowledge about place value while *engaging* in SMPs. After the lesson was over, they debriefed and PSTs shared evidence of different SMPs throughout the activity.

Because of our focus on having our PSTs apply and grow their understanding of the SMPs in a field placement setting, we are framing this study through Kolb's (1984) cycle of experiential learning. PSTs build knowledge about the mathematical practices in their methods course through a variety of experiences including a deep dive into the SMPs and identifying the SMPs while watching videos of elementary math lessons. They have the opportunity to apply this knowledge during the *concrete experience* of their field placement, where they are immersed in an elementary classroom. While the PSTs observe math lessons, they are engaged in *reflective observation*, where they notice and connect what they are seeing the students and teacher do during math instruction to the mathematical practices. As they make sense of the mathematical practices, both during their field placement and during the debriefing process back in their university classroom, the PSTs refine their understanding of the mathematical practices and begin to solidify how they *conceptualize* them. This experience lays the foundation for continued application of the mathematical practices during *active experimentation*, when PSTs use their knowledge of the SMPs to develop problem-based lessons and facilitate the SMPs when teaching in their future field placements, including student teaching.

Research Design

When PSTs go into their elementary field placements, they observe at least two mathematics lessons. The PSTs are asked to describe and identify when they observe the elementary students engaging in the SMPs and when they observe the classroom teacher facilitating learning experiences that allow students to engage in the SMPs, which they record in a graphic organizer. The graphic organizer consists of a 9 x 3 table with the eight SMPs listed in the first column, the last two columns have the headings "student engagement" and "teacher facilitation," and blank cells for PSTs to record their observations. PSTs are expected to include a picture and/or describe the portion of the lesson in words as to what happened that indicated that SMP. When the PSTs return from their field placement, we debrief and they talk about what they saw, including evidence of a given SMP. PSTs are then expected to reflect on their learning about the mathematical practices. For this study we will be focusing on the observations PSTs collected during their field placements.

Participants

Data was collected at two universities in an elementary math methods course. A total of 39 participants were included in the study, 23 from University A and 16 from University B.

Students from University A were second-semester juniors in their first of two math methods courses and their field experience was conducted toward the end of the semester in K-2 classrooms. Students from University B were first-semester seniors taking their only math methods course and their field experience was distributed in two-four week intervals throughout the semester; the assignment being analyzed for this proceeding took place in the third and fourth week of the semester in K-5 classrooms.

Data Collection and Analysis

Data from University A was collected during Fall 2019 and data from University B was collected during Spring 2020. Data consisted of the SMP graphic organizer completed by the PSTs during at least two observations of mathematics lessons in their field placements. Student engagement (SE) corresponds to how our PSTs indicated elementary students were engaging in the SMPs. Teacher facilitation (TF) corresponds to how PSTs' supervising teachers facilitated learning experiences for the elementary students to engage in the SMPs.

Two researchers, who were also instructors of the course, developed the rubric in Table 1 through an open-coding process using the PSTs' descriptions from their SMP graphic organizer to evaluate how PSTs identified the SMPs during their field placement observations. Table 1 also contains an exemplar for each level of the rubric rating for illustrative purposes using MP3 *Construct viable arguments and critiquing the reasoning of others*. We used examples from the "teacher facilitation" cell in the graphic organizer to maintain consistency.

Table 1

Rubric for SMP Graphic Organizer

	Proficient (4)	***Approaching (3)***	***Developing (2)***	***Beginning (1)***	***Blank (0)***
Descriptor	Observation reflects the mathematical practice	Observation is related to the mathematical practice, but displays a common misconception about that practice	Observation is related to the mathematical practice, but does not contain evidence of understanding the practice	Observation is not related to the mathematical practice	This portion of the graphic organizer was blank.
Exemplar of MP3 TF	*The teacher asked, "How do you know your tower has more than 5 blocks?"*	*The teacher asked, "Do you agree with that idea?*	*Ask other students for feedback.*	*The teacher is modeling decimals to students.*	*No response/blank cell.*

In order to ensure inter-rater reliability, the researchers analyzed the data together. If their ratings differed, the scores were discussed using the rubric descriptors, and for every instance they came to the same conclusion. Data from one university was analyzed first; then data from the second university was analyzed using the rubric in Table 1. Even though the assignment was implemented differently at each university, similar misconceptions and themes were found in both data sets. After scoring the responses in the graphic organizer using the rubric, frequencies were found to determine any trends in the data.

Findings

The researchers found frequencies of each of the 16 coded items. The columns are identified by the SMP and whether it was the "student engagement" (SE) cell or the "teacher facilitation" (TF) cell. For example, MP1 SE means we are coding the cell for MP1 and looking at the example of "student engagement." Table 2 identifies the frequency to which each code (0, 1, 2, 3, 4) was identified using the rubric in Table 1 for all 39 participants. Some students completed the student engagement but not teacher facilitation for a particular SMP or vice versa so we kept these separate. We are using Table 2 to identify trends in the data.

Table 2

Rubric Ratings by Mathematical Practice

Rubric Rating	MP1 SE	MP1 TF	MP2 SE	MP2 TF	MP3 SE	MP3 TF	MP4 SE	MP4 TF	MP5 SE	MP5 TF	MP6 SE	MP6 TF	MP7 SE	MP7 TF	MP8 SE	MP8 TF
0	5	6	11	15	10	11	9	9	5	6	11	10	15	18	13	18
1	20	15	15	12	6	7	14	15	4	8	6	8	9	10	5	8
2	3	3	4	5	2	1	2	1	6	3	3	2	4	3	6	4
3	3	11	2	5	12	13	5	11	23	19	2	3	5	6	10	8
4	8	4	7	2	9	7	9	3	1	3	17	16	6	2	5	1

In looking at the overall data, our PSTs most frequently identified MP6 *Attend to precision* at a proficient level (score of 4: SE=17, TF=16). Many PSTs used examples of precise vocabulary and precise answers, including appropriate units. A misconception PSTs had about this SMP was that they identified precision in non-mathematical situations, like writing numerals in the proper format. What we found interesting was the dichotomy in scores for precision; just as many PSTs could not identify it (score of 0: SE=11, TF=10) or misidentified it (score of 1: SE=6, TF=8).

Data also indicated that the following SMPs had a high number of misconceptions (score of 3) compared to other practices: MP5 *Use appropriate tools strategically* (SE=23, TF=19); MP8

Look for and express regularity in repeated reasoning (SE=10, TF=8); and MP3 (SE=12, TF=13). Through teaching and delivering professional development, the researchers have observed that current students and inservice teachers frequently misidentify MP5. Although a manipulative or tool is being used in many of these cases, only *one* tool is being used and it is often selected by the teacher. For students to engage in MP5, the teacher should provide a variety of tools and the *student chooses* which one they want to use. An example of this misconception was noted in one PSTs' work when they said, "Teacher gave cubes to each table" under TF.

MP7 *Look for and make use of structure* and MP8 are so closely intertwined that they are commonly confused. We found that same misconception with our PSTs. For instance, a PST stated that SE in MP8 was when students could "Recognize the 6+1 is the same as 1+6," but this idea of the Commutative Property reflects student engagement in MP7 instead. Another misconception we noted for MP8 was that some PSTs talked about building understanding through doing problems over and over, like timed tests, to memorize answers as opposed to generalizing. Another PST stated, "They understand the addition and subtraction patterns and have memorized the addition facts," in SE, which reflects this focus on memorization.

From Table 2, we also noticed that MP1 *Make sense of problems and persevere in solving them* had the highest rate of being misidentified (score of 1: SE=20, TF=15). This indicates that PSTs did not have an understanding of what activity would be classified as MP1. For instance, another PST stated that an example of SE was "Students are asked to color in the blocks on a paper to match how many blocks they have." Similar to other SMPs, there were also a high number of misconceptions about how teachers facilitate MP1 (score of 3: TF=11. The PSTs misidentified instances when the *teacher* was engaging in MP1 by making sense of the problem and/or persevering in solving it rather than *facilitating* a learning experience that allowed the *elementary students* to engage in MP1. A PST illustrates this misconception when they said, "When student gets it wrong, teacher helps student walk through it without telling them the answer." In this example, the classroom teacher is directing the student to persevere.

Discussion and Conclusion

The findings from this study indicate that developing a deep understanding of the SMPs takes time and that MTEs must be strategic in the way in which they prepare PSTs to understand and identify the mathematical practices. This is not surprising given that many of the misconceptions that arose in this study are similar to the misconceptions the researchers have observed when working with inservice teachers. Even though there are challenges, the authentic experience of seeing the practices in action in elementary classrooms as part of a field placement did allow our PSTs to engage in *reflective observation* and solidify their *conceptualization* of the SMPs through the process of critical reflection (Kolb, 1984). We know that our PSTs will need continued opportunities to refine their understanding of how students engage in and teachers facilitate the SMPs both in our university courses and their future field placements. These opportunities might include generating examples and non-examples for each SMP, continuing to explore videos and/or vignettes of mathematics lessons through the lens of student engagement in and teacher facilitation of the SMPs after field experience, and being explicit about the SMP misconceptions. This will prepare them to do the work of mathematics teachers by developing the mathematical practices in their students (NGA Center & CCSSO, 2010).

While found that our PSTs have a range of understanding an ability in terms of identifying the SMPs, we believe that the misconceptions we found among PSTs as we answered our second research question have greater implications for our work as MTEs. As such, this has made us aware of adjustments we need to make, as MTEs, in our university courses when introducing the SMPs. For instance, when discussing the SMPs in class prior to their placements, we will be more explicit with MP5. For MP5, we need PSTs to understand that students should be given a choice of tools and *students* are choosing what tool they want to use. Many PSTs identified MP5 whenever students were given a manipulative. Additionally, we need to give more examples and non-examples of MP7 and MP8 when we introduce the SMPs, so students can see differences and clearly delineate between these two practices.

Another common misconception we noticed across that SMPs was that our PSTs misinterpreted teacher facilitation as meaning the teacher engages in the mathematical practice itself. However, teacher facilitation of the SMPs is the way in which teachers create learning experiences that support *students* engaging in the SMPs. Therefore, we need to be more intentional with how we describe teacher facilitation in mathematics teaching and support our PSTs in knowing what facilitation of the mathematical practices looks like and sounds like. One aspect of this study that is beyond our control is the type of mathematics classroom in which our PSTs are placed for their field experience. If they are placed in a teacher-directed classroom, we might expect to see fewer instances of teachers facilitating the SMPs and greater instances of misinterpreting what teacher facilitation of the SMPs looks like. However, even in a student-centered mathematics classroom, there could be instances of inservice teachers misinterpreting teacher facilitation as engaging in the SMPs *themselves,* instead of creating opportunities for *students* to engage in the SMPs. As such, it is critical that MTEs explicitly address this misconception with their PSTs by providing examples and non-examples of teacher facilitation of the SMPs, including asking them to identify when they observe instances of this misconception. Experiences like this can support PSTs in developing a deeper understanding of the SMPs as well as prepare them to plan and facilitate mathematics lessons that effectively focus on the practices.

Upon reflection, we also realized there were some logistical issues with the assignment that we need to address in the future. We did not require PSTs to provide videos as evidence for the implemented SMP due to privacy issues with the elementary students in the field placement, but Graybeal (2013) indicated that having the video clip helped determine if the SMP was evident in the lesson. In the next iteration of this assignment, PSTs will be required to elaborate more on their evidence in hopes that there will be fewer scores of 2 when we code data, which indicates we did not have enough detail to determine if the PSTs truly understood the practice. Likewise, if our PSTs do not observe a mathematical practice in the mathematics lesson or if they observe a misinterpretation of that practice as part of teacher facilitation, we will ask them to note that in their graphic organizer. For our next steps, we plan to implement this assignment again with the changes noted above and continue to collect data.

In conclusion, we hope that these findings can inform other MTEs of potential challenges and opportunities as they work to develop a deep understanding of the Standards for Mathematical Practice in their PSTs.

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EXPLORING PERSPECTIVES OF PRESERVICE TEACHERS' ENACTMENT OF TEACHING MOVES

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This paper explores teaching moves enacted by an individual preservice teacher during a one-on one problem-solving interview and the rationales shared for making those teaching moves during a stimulated-recall interview. Relationships between the preservice teacher's teaching moves and rationales revealed alignment and misalignment with the intended goal(s) of teaching moves established in the literature. Recommendations are made for teacher educators to support preservice teachers' development using an asset-based approach and highlights challenges of learning to teach responsively.

The narrative of what it means to be teachers of mathematics is changing and continues to develop. Historically in mathematics education, teachers have been sole keepers of knowledge in the classroom—taking on the majority of the mathematical work while children listen and follow. In contrast, policy documents have emphasized teachers as facilitators who elicit and build on children's mathematical thinking, (CMT) by listening closely in-the-moment and being responsive to their needs (Linewand et al., 2014). Responsive teaching is one type of teaching that embodies this narrative.

Responsive Teaching

Although there are many ways teachers can be responsive to children in the classroom, I adopt Robertson's et al. (2016) conceptualization of responsive teaching and apply it to mathematics—one that requires teachers to attend to the details of CMT, maintain the focus on the underlying mathematical concepts, and employ opportunities to take up children's ideas and follow them. This type of teaching positions children as capable and having assets on which teachers can build. Further, responsive teachers aim to better understand CMT through questioning their ideas—not evaluating them—and use what they learn to make instructional decisions in-the-moment (Jacobs & Empson, 2016). To capture the complex nature of questioning CMT, I use the term teaching moves. *Teaching moves* include questions, statements, or even actions (Jacobs & Empson, 2016). The ways in which teachers enact these instructional practices prompt children to respond in different ways, which can impact what children learn and what teachers understand about CMT.

Eliciting and Building on Children's Mathematical Thinking

Researchers use a wide range of terminology in their discussion of teaching moves that elicit and build on CMT (and those that do not). Thus, to synthesize these ideas, I formed categories of teaching moves—composed of various forms—based on the intended goal of each teaching move established in the literature. Although multiple categories of teaching moves exist, for the purpose of this paper, I focus on three more prevalent categories, (a) comprehending story problems (b) exploring details of children's mathematical thinking, and (c) telling information to children. Further, since this study focused on one-on-one conversations between a teacher and a child, I did not include teaching moves geared toward whole class discussions such as revoicing (see e.g., Franke et al., 2015). First, comprehending story problems is a category of teaching moves that supports children in understanding the story situation and using stories as tools for sense-making (Ball, 1993, Jacobs & Empson, 2016). Second, exploring details of CMT is a category of teaching moves in which teachers focus on the mathematical details of what children say and do. For instance, teachers may invite children to share how they solved or press for reasoning about a detail to gain additional insight (Franke et al., 2015; Jacobs & Empson, 2016; Shaughnessy & Boerst, 2018). Third, telling information to children is a category of teaching moves that provides children with ideas teachers believe important for problem-solving such as demonstrating for children what to do or naming of terminology (Cengiz et al., 2011; Moyer & Milewicz, 2002; Sun & van Es, 2015). In sum, various teaching moves have the potential to be responsive, depending on how they are enacted.

Developing Expertise in Responsive Teaching

Research has shown a variety of ways practicing teachers and PSTs develop expertise in responsive teaching such as use of lesson sketch, simulations, and rehearsals (Grossman et al., 2009; Shaughnessy & Boerst, 2018; Webel et el., 2018). Two commonly used ways to develop expertise in responsive teaching include videos of teaching and engagement in problem-solving interviews. Videos allow teachers to see and hear CMT and can be re-played as needed. Problem-solving interviews provide opportunities for teachers to engage directly with CMT and practice their questioning in a "low-risk" setting. In short, there is a growing research base about how teachers elicit and build on CMT as well as how this expertise develops.

Current Study

Much of what we know about the way teachers elicit and build on CMT come from practicing teachers. However, we know PSTs use similar categories of teaching moves with less expertise as studies often compare their skillsets to those of practicing teachers or evaluate where support is needed (Jacobs et al., 2010; Sun & van Es, 2015; Webel et al., 2018). Moreover, voices of PSTs—rationales underlying the teaching moves enacted—are often not foregrounded in this research. Similar to research that describes the importance of teachers being responsive to CMT, this study is built on the assumption that it is important for teacher educators to be responsive to the thinking of PSTs. Teacher educators cannot be responsive to PSTs' thinking if this thinking is not elicited. The thinking of PSTs, or the underlying reasons for their decision-making is referred to in this study as *rationales*. Therefore, this study was designed to add to the literature on responsive teaching including teaching moves PSTs make, rationales they provide for those teaching moves and to understand the relationship between their rationales and the teaching moves used when working with children. For the purposes of this paper, I focus on a research question from a larger study, *what is the relationship between PSTs' teaching moves and their rationales for making them?*

Methods

In the larger study, a mixed-methods design was used to better understand the teaching moves PSTs enacted and their rationales for making them in order to explore the relationships between teaching moves and rationales (Smithey, 2020). Further, this study aimed to gain a sense of how PSTs naturally engaged with children prior to explicit instruction in teaching mathematics. Thus, PSTs enrolled in the study had not yet taken a mathematics methods course.

Upon recruitment, a total of 11 PSTs volunteered to be part of the study. For this paper, I focus on one PST, Julianne (a pseudonym), because she best represents the range of relationships between teaching moves and PSTs' rationales found within the larger study. Julianne, a Caucasian female in her early twenties, was in the first semester of a two-year education licensure program located in the southeastern region of the United States. Prior to this semester, Julianne had taken one mathematics K-6 content course as well as an introduction to education course where she observed a few hours a week in the elementary classroom.

Data Sources

As part of the larger study, Julianne was recruited to participate in two interviews, a problem-solving interview (PSI) and a stimulated-recall interview (SRI). A PSI is a one-on-one conversation between the PST and a second-grader around a series of mathematical story problems. The purpose of this PSI was to capture teaching moves enacted with children. The PSI included story problems that were designed to include similar content (whole number), and strategic selection of contexts and problem structures. This interview lasted approximately 15 minutes and was audio and video recorded. Second, the SRI provided space for PSTs to playback the video of their PSI shortly after it ended to retroactively recall the rationales for their teaching moves. After discussing the PST-selected teaching moves, teaching moves not yet discussed were revisited, and I asked about their rationales in similar ways. The SRI lasted approximately 45 minutes and was audio and video recorded to be able to explore relationships during analysis.

Data Analysis

Data was analyzed in multiple phases. In *Phases 1 and 2,* PSIs and SRIs were explored separately and iteratively through qualitative and quantitative analyses with the goal of developing and applying coding schemes to capture teaching moves used and the rationales for making them. Development of coding schemes for the teaching moves began with a list of teaching moves from the literature and through constant-comparative analysis. It is important to note the teaching moves were coded as executed, not coded based on the PSTs' intention nor the quality of enactment. In contrast, the coding scheme for the rationales were derived using grounded theory from the SRIs—to honor the voices of the PSTs. For the purpose of this paper, I focus on *Phase 3* of data analysis where I connected qualitative and quantitative findings from Phase 1 and 2. The goal of Phase 3 was to explore relationships between teaching moves and rationales at the categorical level. To analyze the relationships, I created matrices to compare the categories, relative frequencies, and whether the rationales aligned or misaligned with the ideal goal of each teaching-move category—determined from my perspective, (informed by the literature), on the ideal goal of each teaching-move category. Note that each teaching-move category includes a variety of forms (e.g., inviting to share or pressing for reasoning), but for this analysis, as long as one of the forms of that teaching-move category aligned with the rationale category, that pair was considered aligned. In the following section, I use Julianne's data as it is representative of the relationships found in the larger study (Smithey, 2020).

Results

Julianne enacted a total of 44 teaching moves across three categories, (a) exploring details of CMT, (b) comprehending story problems, and (c) telling information to children. Of the 44 teaching moves, Julianne discussed 31 of them during the SRI which linked to 40 rationales. Further, the number of rationales is greater than the number of teaching moves because sometimes Julianne had more than one rationale for enacting a teaching move. Although the larger study included multiple categories of rationales, Julianne's rationales fell into three categories, (a) enhancing children's understanding, (b) guiding children's problem-solving, and (c) enhancing PSTs' understanding. First, enhancing children's understanding is a category of rationales in which Julianne wanted the child to better understand their strategy, the context, or a mathematical idea. Second, guiding children's problem-solving is a category of rationales in which she prioritized the child getting to the answer *over* enhancing their understanding. Third, enhancing PSTs' understanding is a category of rationales in which Julianne wanted to expand, confirm, or develop her own understanding of the child's thinking. In sum, the rationales were explanations provided for making the teaching moves she did, not the teaching moves themselves.

In comparing the ideal goal of the teaching move and Julianne's own rationale for making a teaching move at the categorical level, sometimes those goals aligned and other times they did not. Although this would be expected for PSTs, upon closer examination these relationships provide specific insight into the capabilities Julianne has as well as the challenges she may face in learning to teach responsively. To best illustrate the complexity of these relationships, I focus here on one teaching-move category, exploring details of CMT. Across Julianne's PSI, 19 teaching moves explored details of CMT, and she shared 25 rationales for making those teaching moves. Julianne showed evidence that her rationales for exploring details of CMT *aligned* with the intended goal of providing space for children to share the details of their strategies. Rationales that aligned with the teaching-move category, focused on enhancing the child's understanding as well as her own understanding, which collectively made up 72% of the total number of rationales for exploring details of CMT. In contrast, the rationale category of guiding children's problem solving (prioritizing arriving at the answer) misaligned with the intended goal of providing space for children to share details of their strategies, which made up 28% of the total number of rationales Julianne provided for exploring details of CMT.

More specifically, Julianne's rationales for enacting teaching moves that explored the details of CMT aligned when she discussed the desire for the child to understand the details in their strategy, the story problem context, or the desire to develop her own personal understanding of what the child was thinking. In contrast, some of Julianne's rationales misaligned with the teaching-move category of exploring details of CMT as she indicated a priority in helping the child get to the answer yet used teaching moves intended to give space for the child to share their thinking independent from the teacher. Although we want to encourage PSTs to explore the details of CMT, the rationale for doing so should lie in learning more about the details of children's strategies *not* in arriving at an answer.

To further illustrate this alignment and misalignment, a story problem from Julianne's PSI is used: *Deja had 33 buttons. She put the buttons into 3 bags with the same number of buttons in each bag. How many buttons did she put in each bag?* To provide an overview of the child's strategy, the child chose the hundreds chart and shared that she was going to take away 3 at a time until she had taken away 33. However, when she started counting, she *counted up* 3 at a time, starting at 33. When the child got to the end of the chart (100), she used base-ten blocks to continue counting up by threes. After Julianne encouraged a change of strategy, the child organized the cubes into 3 uneven groups and spent the rest of the time counting and recounting individual piles and the whole set.

At the beginning of the button problem, Julianne invited the child to share her problem-solving plans (exploring details of CMT) prior to solving. Julianne's rationale for enacting this teaching move aligned because Julianne was trying to enhance her own understanding of how the child was thinking as seen in Table 1.

Table 1

Example of Alignment: Exploring Details of CMT and Enhancing PSTs' Understanding

Problem-Solving Interview	**Stimulated-Recall Interview**
So what are you going to do there? *(teaching-move category of exploring details of CMT)*	Because I really, I wanted to, again know the process she was going to do before she did it so if there was a chance that I need to stop her [and explain] this is what you're trying to do. Because if I went into it blank, I wouldn't know what she was doing to be able to help her. *(rationale category of enhancing PSTs' understanding)*

As the interaction continued, the child began her strategy by counting up from 33, by threes, on the hundreds chart. After the child had counted past 50 on the hundreds chart, Julianne interrupted and explored the details of the child's thinking by pressing for more information about her strategy. Julianne's rationale for enacting this move was twofold (see Table 2). The first part aligned with providing space for the child to share reasoning about their strategy because Julianne was confused about what the child was doing. Thus, she tried to enhance her own understanding of CMT. However, the second part of the Julianne's rationale was misaligned with providing space for the child to reason about her strategy because the focus was trying to guide the child's problem solving or, in Julianne's words, "re-route her."

Table 2

Alignment and Misalignment of Rationales: Exploring Details of CMT

Problem-Solving Interview	Stimulated-Recall Interview
Explain to me what you are doing right now. *(teaching-move category of exploring details of CMT)*	This is one of those places where I stopped her because I did not understand how she was going all the way into the 50s. And I wanted to see where she was *(rationale category of enhancing PSTs' understanding)* and if there was any way I could re-route her and to, I guess one of the correct ways of doing it. *(rationale category of guiding children's problem solving)*

These examples from this story problem, showcased how Julianne used teaching moves to explore the details of the child's mathematical thinking and how at times her rationales aligned (and misaligned) with providing space for the child to share their thinking.

Discussion

The goal for examining teaching moves and rationales together was to understand the relationships that may exist between them and what that means in the development of PSTs as responsive teachers in mathematics. In this case, we were able to better understand the rationales Julianne had that aligned with the goals of the teaching moves she enacted but also see instances when Julianne used teaching moves that did not align with her rationale. Furthermore, we were able to see instances Julianne used a teaching move that was responsive to CMT but from her perspective, the goal was for the child to arrive at an answer—proving more important than the child understanding. By asking PSTs to reflect on their practice in more specific ways, teacher educators have opportunities to listen and learn from PSTs—identifying the assets they bring to the classroom as well as the challenges they may face in learning how to teach in responsively.

I argue teacher educators should refine the ways they ask PSTs to reflect on their practice. Typically, after teaching experiences, we ask PSTs to reflect on how they felt, what they learned, or what they might have done differently (see e.g., Webel et al., 2018). Although reflecting in these ways is well documented as an effective learning tool, PSTs' *recalling* their in-the-moment decision making, as they did in the SRIs, would be an additional tool for learning (Smithey, 2020). Further, asking PSTs to reflect on why they used particular teaching moves and noting how children respond can help PSTs better align their teaching moves with their goals.

As the field continues to encourage being responsive to the needs of children (Robertson et al., 2016), I argue we should extend the notion of responsive teaching to teacher education and aim to draw on the assets of those we teach. Further, teacher educators can elevate the perspectives of PSTs to inform our instructional decisions. Finally, I urge the field to consider valuing voices of those we teach and the power in using their perspectives in our research.

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Reestablishing Connections for Elementary Students

FOURTH-GRADE STUDENTS' SENSEMAKING OF WORD PROBLEMS

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The purpose of this study was to investigate fourth-grade students' sensemaking of a word problem. Sensemaking occurs when students connect their understanding of a situation with existing knowledge. We investigated students' sensemaking about a word problem by comparing students' strategy use. Inductive analysis was used to find themes about student sensemaking. Students exhibited one of three levels of sensemaking. Some problem-solving strategies, as a result of students' sensemaking, led to a greater frequency of correct results.

Standards represent each states' expectations for what content should be taught. Many states have adopted some form of the Common Core State Standards for Mathematics (CCSSM; CCSSI, 2010). The CCSSM established real-life problem solving as something students should be engaged in throughout their K-12 education (CCSSI, 2010, p. 6, 7, 84). Furthermore, teachers should promote students' mathematical proficiency through providing opportunities for students to "make sense of problems and persevere in solving them" (CCSSI, 2010, p. 6). This study investigates fourth-grade students' sensemaking about a *multi-step situational word problem*, providing the mathematics education community with evidence about students' sensemaking in the Common Core Era.

Theoretical Frameworks: Problem-solving and Sensemaking

This study is framed by notions of problem solving and sensemaking about situational word problems. Broadly speaking, problem solving "is what you do when you don't know what to do" (Sowder, 1985, p. 141). Verschaffel et al. (2000) describes a six-stage model of problem solving that includes (a) reading the problem, (b) creating a representation of the situation, (c) constructing a mathematical representation of the situation, (d) arriving at a result from employing a procedure on the representation, (e) interpreting the result in light of the situational representation [see (b)], and finally, (f) reporting the solution within the problem's context. In consideration of students' sensemaking, we utilize a framework for problems such that the word problems are (a) open, (b) developmentally complex, and (c) realistic tasks for an individual (Verschaffel et al., 1999). Open tasks can be solved using multiple developmentally appropriate strategies. Word problems therefore are mathematical tasks presented as text, which contain real-life situational background information (Verschaffel et al., 2000). We define strategy as the mathematical pathway an individual enacts while problem solving, which includes both representations and mathematical procedures (Goldin, 2002).

Sensemaking about Word Problems

Sensemaking is when students develop an understanding of a situation or context by connecting it with existing knowledge (NCTM, 2009, p. 4). The way students make sense of problems can vary quite a bit due to cognitive, social, and environmental factors (Cifarelli & Cai, 2005). During problem solving, students need to make sense of the word problem by observing connections between the situation being presented and the mathematical representations and operations necessary for a solution (Verschaffel et al., 1999; Verschaffel et al., 2009). The word problem increases in sensemaking difficulty when the situation necessitates more than one operation, and the use of the result from the previous operation must be interpreted and used in the context of a different operation (Quintero, 1983). Sensemaking is essential for successful problem solving (Pape, 2004; Verschaffel et al., 2000). Development of sensemaking habits help students develop autonomy, relying on their own reasoning and resources to be more persistent while problem solving (Meuller et al., 2011; Yackel & Cobb, 1996), and ultimately foster productive dispositions as mathematically proficient problem solvers.

Sensemaking occurs at many steps in the problem-solving process (Verschaffel et al., 2009) and some have focused on students' work between the situation and mathematical stages as a way to explore sensemaking. For instance, Palm's (2008) qualitative study examining fifth-grade students' work indicated that students' engagement with realistic word problems increased the likelihood their problem solving ended with a correct solution to a problem. Similarly, Yee and Bostic (2014) also conducted a qualitative study examining secondary students' word problem solving and drew a conclusion that more successful problem solvers were flexible with their mathematical representations often using non-symbolic representations, compared to others who employed symbolic tools. Taken collectively, the literature provides ideas about students' problem solving but few take a critical look at students' work to explore their mathematical sensemaking of word problems. Hence, this study aims to fill a needed gap within the problem-solving literature.

Method

The Fair Task

This study stems from a broader grant-funded project aiming to develop problem-solving tests that align with the Common Core State Standards for Mathematics in grades 3-6. Each Problem-Solving Measure (PSM) is composed of 15 items addressing grade-level content. Validity evidence has been gathered for each test and led to a robust and valid score interpretation and use arguments (e.g., Bostic et al., 2019). In this study, we investigated students' sensemaking of one purposefully selected word problem from the PSM for grade 4. The Fair Task states, "Josephine sold tickets to the fair. She collected a total of $1,302 from the tickets she sold. $630 came from the adult ticket sales. Each adult ticket costs $18. Each child ticket costs $14. How many child tickets did she sell?" It incorporates multi-step thinking and addresses Operations and Algebraic Thinking (OA) standards. Specifically, students are expected to make sense of a mathematical difference and the number of groups within it. This task was selected because (a) it is of moderate psychometric difficulty for average-performing students, (b) multiple developmentally-appropriate strategies have been used to solve it, and (c) it is connected to standards that are linked with fostering algebraic understanding (Smith, 2014). Through the PSM validation process, the Fair Task was reviewed by mathematicians, mathematics educators, and mathematics teachers. Drawing upon the knowledge of these experts three key observations (KO) to successfully solve the Fair Task were generated. These KOs are tied to sensemaking of various parts within the word problem. (KO1) The difference between $1,302 and $630 is the dollar amount brought in by selling child tickets. This value is $672. (KO2) Each child's ticket is $14. There is some number of groups of 14 that represent the number of child tickets sold. (KO3) The number of groups of 14 within the unit of 672 indicates the number of child tickets sold. We drew upon these KOs to explore two research questions. (RQ1) How do students draw upon sensemaking while solving the Fair Task? (RQ2) What mathematical strategies did students use while problem solving and how were those strategies related to students' successful problem solving on the Fair Task?

Participants and Setting

In total, 280 fourth-grade participants were included in the study. They came from a rural and a suburban school district located in a Midwest state that adopted the CCSSM. The PSM4 was administered near the end of the academic year in paper-and-pencil format. PSM4 administration followed the same practice as usual. Students solved problems individually, in a quiet classroom setting monitored by the researchers and a classroom teacher. They did not use calculators, had up to 120 minutes for test administration, and were encouraged to write, draw, and represent their ideas on the testing paper. Any students named in this proceeding are pseudonyms.

Data Collection and Data Analysis

Participants solved the Fair Task and expressed their strategy use and result from problem solving in writing. The written work on the Fair Task was reviewed by a team of three researchers. This largely qualitative study of students' written mathematical work on the Fair Task used inductive analysis (Hatch, 2002) to generate themes about students' sensemaking. The coding process of analysis had multiple steps. Three researchers read the solutions of all 280 students. The frame for the analysis was evidence of student mathematical sensemaking of the problem related to the three key observations for the Fair Task. Researchers looked for sensemaking as evidenced by student work conveying understanding of the connection between the Fair Task context and students' mathematical strategies for solving the problem at hand. The researchers identified salient domains, clusters of strategy types, and gave them a code. Each researcher took a specific domain and reread all of the students' solutions to decide if the domains were supported by the data. Discrepancies were shared with the research team and discussed for consensus. This completed analysis for RQ1. The authors created written paragraphs and graphic maps to describe each domain. The completed domains were analyzed, within and across, for patterns involving students' solution strategies for the Fair Task. When patterns among student strategies were found, further analysis on the participants work exhibiting those patterns was conducted to determine the level of success among the strategies used. This completed the analysis for RQ2. Data excerpts to support the patterns are shared.

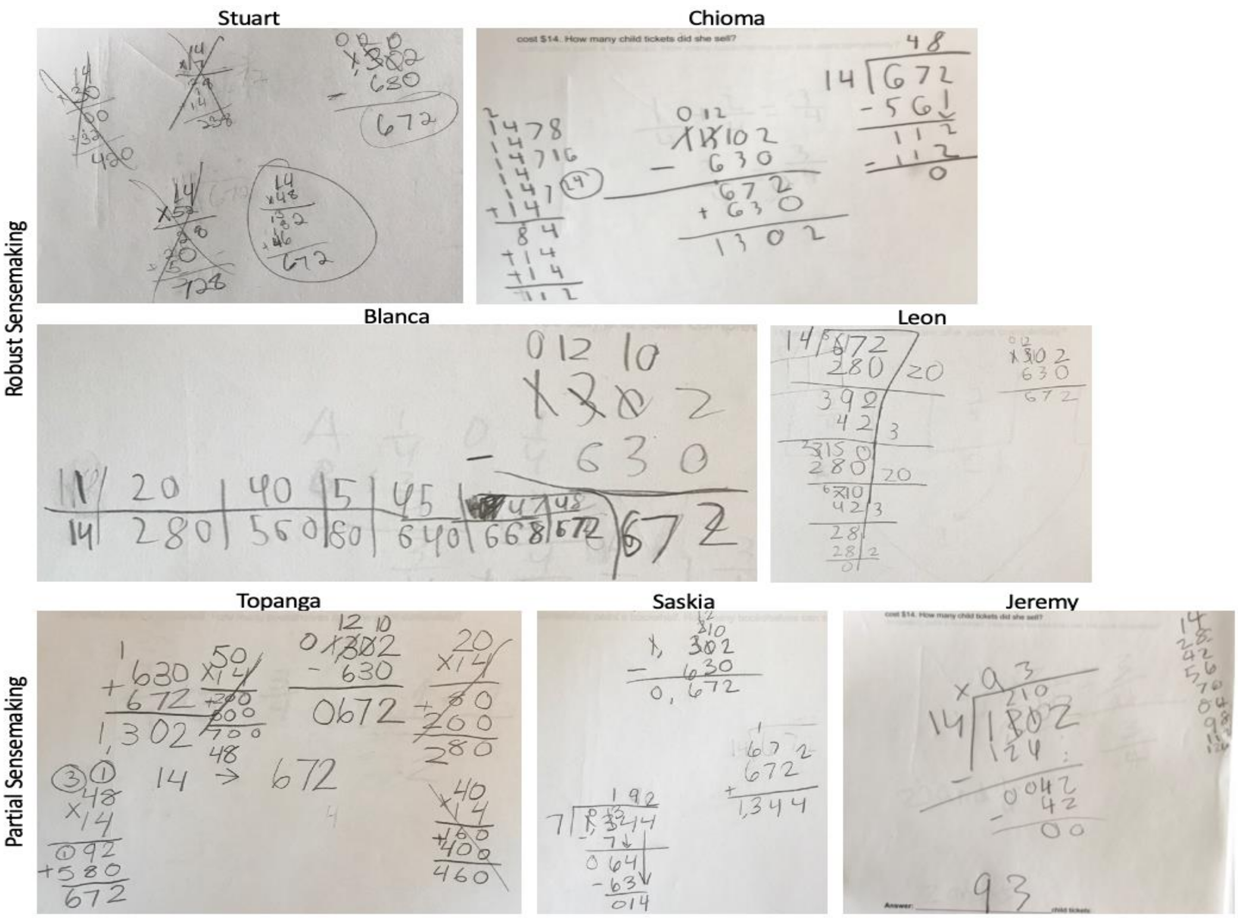
Findings

RQ1: Sensemaking of the Fair Task

Inductive analysis revealed three qualitatively different levels of student sensemaking. These domains were labelled as: robust evidence of sensemaking, partial evidence of sensemaking, and no evidence of sensemaking. *Robust evidence of sensemaking* about the Fair Task indicated attention to all three key observations necessary to solve the problem. Seventy-nine of the 280 students (28%) in our sample provided evidence that they made sense of the key observations and enacted 14 unique mathematical strategies to derive an answer. While strategies varied among the 79 students, 50 students arrived at the correct answer. The remaining 29 students had evidence of their sensemaking about all three KOs, but didn't arrive at the solution due to a minor arithmetic error. This suggests that generally speaking, students who made sense of the difference, the number of groups, and the number of groups within the appropriate difference, arrived at the appropriate solution. Figure 1 offers four samples of student work evidencing robust sensemaking through different strategies.

Figure 1

Student Samples for Robust and Partial Sensemaking



Note. Student work samples of different strategies for robust and partial sensemaking of the Key Observations needed to solve the Fair Task.

Some students in our sample demonstrated *partial evidence of sensemaking* about the Fair Task through their attention to mathematical work for KO1, KO2, or both KO1 and KO2, but did not provide evidence for KO3. One hundred fifteen of the 280 students (41%) provided evidence that they made sense of either the difference, the number of groups of 14, or both. However, these students were unable to demonstrate evidence of their understanding for KO3. This is depicted in the examples in Figure 1. The students in this domain exhibited eight mathematically different strategies.

Students' work lacking evidence for any of the three key observations were classified as *no evidence of sensemaking*. Eighty-six of the 280 students (31%) provided no evidence that they had made sense of any of the three key observations. Broadly speaking, students in this domain either enacted strategies that did not lead to a correct solution of the Fair Task or gave no evidence of how they arrived at their solution.

RQ2: Strategic Choices for Finding the Difference (KO1)

As students made sense of KO1 involving the difference between 1302 and 630, they had representational and operational choices to make. Three strategies were identified: (a) Standard Algorithm, which involves a symbolic representation to perform vertical subtraction (b) Adding Up, which involves a symbolic representation of adding up from 630 to arrive at 1302; and (c) Number Line, which involves a pictorial representation of adding up from 630 to arrive at 1302 using a number line. Standard Algorithm was the most prevalent strategy among students as it was used by 159 of 174 students who attended to KO1. Adding Up from 630, a much less prevalent strategy than Standard Algorithm, was used by 14 of 174 students who attended to KO1. Number Line was only used by one student.

RQ2: Strategic Choices for Finding the Number of Groups (KO2)

Students used a variety of methods to find the number of groups of 14 to represent the number of child tickets sold (KO2). Students used both number-based and digit-based operational procedures. The number-based operational procedures that students used included the following: repeated subtraction and multiplication, multiplication using the standard multiplication algorithm or the box method, partial quotients using a traditional or nontraditional setup, and compensation. The digit-based operational procedures that students used included the standard algorithm for division alone or in combination with repeated subtraction, addition, and/or multiplication. Table 1 shows the number of participants in each group including the number of participants who found the correct number of groups of 14 that go into 672. The evidence indicates that students who engaged in the Fair Task were most successful with finding the number of groups when using multiplication. Students were equally likely to be successful when using the standard algorithm and partial quotients to do traditional division and students were only successful 11% of the time when using repeating addition and subtraction.

Table 1

Student Strategies Attending to Key Observation 2

Strategy	Number Based or Digit Based	Total Participants	Number of participants with correct computation	Percentage
Repeated addition and subtraction	Number Based	9	1	11.11%
Multiplication	Number Based	10	10	100%
Standard Division Algorithm	Digit Based	39	26	66.67%
Partial quotients	Number Based	20	12	60%

Summary of Findings

The number of different strategic choices made by students demonstrated the open nature of the Fair Task. The various strategies illuminate differences in students' sensemaking and response to multi-step word problems. Out of the 174 students who gave evidence for making sense of KO1, 159 of them used the standard algorithm for subtraction, including all of the students who showed robust evidence of sensemaking. In contrast, KO2 opens up the pathway for division to be used as the students need to find the number of groups of 14 that go into 672. However, the students in this study used all four operations to make sense of and solve KO2. Students' strategies when sensemaking about KO2 showed that many understood they were able to use properties of operations and the relationships between the operations in their quest to find the number of groups. However, there were differences in the rates of success among the strategies as only two-thirds of the students using algorithmic processes for division proceeded to get the correct answer while 100% of the students using multiplication methods to arrive at the number of groups of 14 arrived at the correct answer. Lastly, only 28% of all students were able to demonstrate sensemaking about the connection between KO1 and KO2 and this greatly restricted the number of students who could successfully solve the problem.

Connections to Literature

The findings here support and extend the current sensemaking literature by examining students' sensemaking in each key observation of a multi-step word problem. Students struggled the most with KO3 (difference and number of groups of 14), which required them to makes sense how the two mathematical ideas connected. This supports Quintero's (1983) assertion about word problems difficulty and Pape's (2004) conclusion that sensemaking is essential for successful problem solving. This study also extends the problem-solving literature (e.g., Palm, 2008; Yee & Bostic, 2014) by providing evidence about which strategies chosen by fourth-grade students tended to yield the most success. Overall, sensemaking and procedural proficiency were revealed to be co-dependent attributes for fourth-graders successful problem solving.

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VISUAL MODELS ON STANDARDIZED TESTS: STUDENTS' MULTIPLE INTERPRETATIONS

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Standardized testing has expanded to incorporate more non-symbolic representations. On the surface the inclusion of more concrete visual models appears to be a productive step in providing a more meaningful assessment of students' understanding. However, this study uses a lens of process/product (Sfard, 1991) to highlight a complication that emerges as students attempt to interpret a model presenting a narrow view of a mathematical concept, specifically a quotient view of fractions.

Introduction

Representations play an important role in the learning and understanding of mathematics (NCTM, 2000). In the domain of rational numbers, visual models support students in developing an understanding of fractions that goes beyond the procedural approach traditionally emphasized where students apply rules and treat the constituent symbolic parts simply as whole numbers (Moss & Case, 1999). In an effort to elevate the significance of models, the Common Core State Standards for Mathematics (CCSSM; CCSSI, 2010) specifically denotes the use of visual fraction models as an essential component to understanding fractional operations. In response, many states' standardized tests have begun to expand their assessments of students' understanding of fractions to include non-symbolic representations. While the inclusion of different representations allows for students to illustrate a broader understanding of fractions as well as elevates their importance, it also brings up other possible complications. Models, like all representations, do not have a uniform interpretation (Gould, 2013; Thompson & Lambdin, 1994). Their meaning depends on what students see and how they interpret the various parts. This is particularly problematic with fractions as their meaning readily shifts depending on what students define as the whole. The goal of this study was to investigate how students navigate the tension that arises when they are assessed on their understanding of a model that can be construed in a variety of ways, but the standardized nature of the test means there is an implied correct interpretation.

Theoretical Perspective

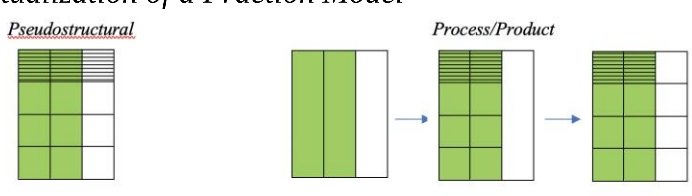
Mathematical concepts can be understood in two different, but related ways: operationally, as processes, and structurally, as objects (Sfard, 1991). For example, even the fundamental notion of whole numbers can be conceptualized in both ways. Initially, children see numbers as the process of counting real world objects. Over time, as students gain experience with counting, this process becomes reified and they begin to see numbers as an object, specifically the result of this counting process, capturing the cardinality of the set of objects.

The ability to interpret representations as embodying both a process and product is a key mathematical understanding (NCTM, 2000). However, all too often students are introduced to powerful mathematical representations without developing an understanding of the underlying mathematical processes they embody (Sfard & Linchevski, 1994). Consequently, they view and act on the representation as an object itself, detached from any operational underpinnings, a conceptualization referred to by Sfard and Linchevski as a *pseudostructural*. Moreover, with such a superficial understanding of the representation, any subsequent processes performed on it seem completely arbitrary, devoid of meaning.

Sfard and Linchevski (1994) explored this conceptualization within the context of algebraic symbols. However, with the use of models expanding in mathematics education, a similar phenomenon seems to have manifested itself with more concrete representations. For example, we see a *pseudostructural* conceptualization characterizing the resulting understanding reported by Webel et al. (2016), where teachers provide students an algorithmic method to apply to a fraction multiplication model. Given the problem $\frac{1}{4} \times \frac{2}{3}$, students are instructed to shade $\frac{2}{3}$ of the rectangle vertically and $\frac{1}{4}$ of the rectangle horizontally (Figure 1). The intersection provides students the answer $\frac{2}{12}$ using a part-whole view of the model, but without having them engage in the process of finding $\frac{1}{4}$ of $\frac{2}{3}$, which is fundamental to understanding fraction multiplication. Alternatively, students applying an operational conceptualization will first draw $\frac{2}{3}$ of the whole and then find $\frac{1}{4}$ of this part. The process of $\frac{1}{4}$ operating on $\frac{2}{3}$ results in two parts and must be interpreted relative to the original whole to arrive at the answer $\frac{2}{12}$.

Figure 1

Pseudostructural Conceptualization of a Fraction Model

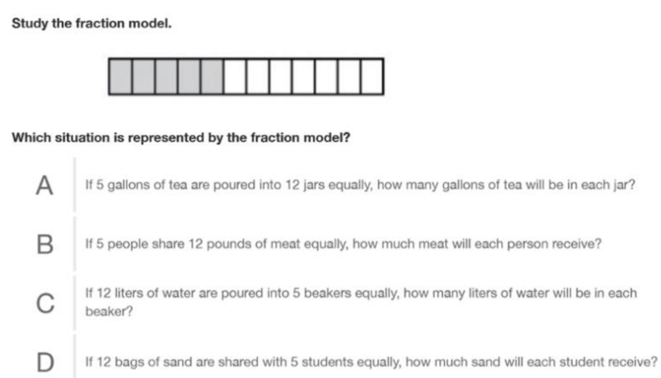


Methods

To explore how students interpret fraction models that appear on standardized tests, we provided a shortened version of the district administered end of unit test to 17 fifth grade students. The test consisted of six multiple choice questions from the previous year, each of which included a model, either in the question or answer, and involved a contextual situation of equal sharing or fraction multiplication. In an attempt to ensure a diversity of thinking, students were selected from three different classes. In addition, a range of students, based on the teachers' perception of aptitude, were invited to participate in the study. After completing the test, we conducted semi-structured interviews with each student. During the interview students were asked to explain how they initially solved the question, followed by a series of questions specifically targeting their understanding of the model. These interviews were video-taped and relevant portions were transcribed. After making detailed notes of all six questions, we focused our analysis on one equal sharing problem (Figure 2) in which students demonstrated a wide range of interpretations of the model presented. The differences that emerged seemed to be rooted in whether the students saw or anticipated the model as 1) representing an object void of any process, 2) a model of the equal sharing process, or 3) both a process and the resulting product. Taking this lens, we analyzed the data once again, and found these three categories served to account for the differences in students' understanding.

Figure 2

Equal Sharing Problem

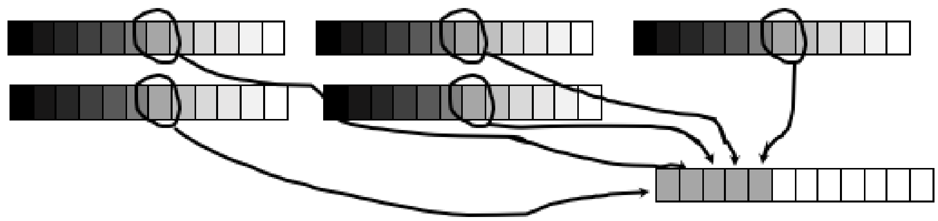


Problem Analysis

The above problem targets a quotient understanding of rational numbers using the context of equal sharing. The correct answer is A: *If 5 gallons of tea are poured into 12 jars equally, how many gallons of tea will be in each jar*. Like all mathematical concepts, rational numbers can be interpreted both as a process, the division of whole numbers, and a product, the fractional result of division. The above contextual problems target this dual understanding, although the model provided only seems to capture the product. Contrast such a model to what students often do when answering such a question. Attempting to capture the process of equal sharing, they draw each of the 5 gallons of tea and then distribute them equally into 12 jars (Empson & Levi, 2011). A common strategy is to divide each of the 5 gallons into 12 equal parts, then allocate 1 part from each gallon to one of the 12 jars.

Figure 3

Representation Capturing the Equal Sharing Process



Each part represents a twelfth of a gallon and the five together result in five twelfths of a gallon that is poured into each jar. By drawing the five gallons and partitioning them up, students capture the process of equal sharing before consolidating the amount together in a resulting product representation. It is the act of physically distributing the original 5 gallons and connecting this action to the final amount in each jar that supports students seeing $\frac{5}{12}$ both as the process of dividing the 5 gallons into 12 jars and the resulting product of $\frac{5}{12}$ of a gallon.

As noted, the model that students are to interpret on the test does not include this process, but only the resulting amount. Students must provide this process themselves and correctly connect it to the model. Such a connection is particularly challenging as students must realize that the whole long rectangle does not represent the 12 jars or even one of the 12 jars, but rather one of the initial gallons of tea of which $\frac{5}{12}$ is poured into one of the jars.

Results

Of the 17 students, 11 did not attempt to conceptualize the equal sharing process within the model. The other six students all engaged in this process but differed in how they connected such understanding to the model provided. Three of these six explained that none of the answer choices connected to the model with two of these students emphatically dismissing all answers because they failed to capture this process. The final three attempted to interpret the model as both a process and the resulting product. While successful, they arrived at two different, yet logical interpretations of the model, and consequently different answer choices.

Category 1: Pseudostructural Understanding

The 11 students who interpreted the model without connecting it to the associated equal sharing process all conveyed a *pseudostructural* understanding. Seven students described the model simply as representing 5 out of 12, a part-whole conceptualization, and interpreted the pieces as whole numbers. For example, Jalen explained that the “overall rectangle represents 5 of something. It is 5 twelfths of anything.” Such a characterization highlights that Jalen viewed the model no differently than the decontextualized symbolic representation of $\frac{5}{12}$. Similarly, Steven summarized his understanding of the model as, “It doesn't really tell you what to do on this one. It’s just saying there are 5 shaded in pieces out of 12...5 over 12.” When pressed to explain why he chose c), he stated that all of the choices could have been correct, but explained that for c) the 5 shaded pieces represented the 5 beakers and the total 12 blocks were the 12 liters of water. Such a construal was the case for all of the students in this category. Regardless of their answer, they saw the model as a decontextualized $\frac{5}{12}$ and associated the 5 shaded squares and the total 12 squares as representing whatever quantities corresponded to 5 and 12 in the problem they chose.

This latter interpretation embodies the understanding demonstrated by the other four students in this category. Rather than seeing the model as a fraction, they simply matched the whole number quantities provided in their selected answer choice with the corresponding whole number of blocks. With such a view of the model, students just resorted to guessing as any of the four choices could be matched as such. Regardless, in both cases, the model provided no meaningful connection to the process.

Category 2: Rejection of Answers-Model Fails to Represent Process

Three students fell in a second category. They focused on an equal sharing process but were unable to connect their reasoning to the model provided. One student, after explaining his understanding, stated that he was unable to see how equal sharing would be represented in the model and thus resorted to guessing. The other two concluded that no answer choice was correct. When asked why, these two students explained that the given model did not capture the equal sharing process that the problems indicated and then offered illustrations of what the model should look like for different answer choices.

To demonstrate their thinking, we turn to Carlota, who not only provided her own alternative representations for answer choice a) and b), but also articulated her interpretation of the given model. She began by explaining, “I didn’t think that any of these were correct answers, because in my mind the fraction model represented subtraction.” She further clarified her reasoning by giving an example of a problem that she believed the model represented. “If you gave 5 pieces of candy to 12 friends, how many pieces would you have left?” She then illustrated the solution of this problem with the model, by drawing the 12 blocks and then shading 5 of them, explaining that this represents the 5 she gave away.

We see Carlota’s explanation as significant because her interpretation of the given model demonstrates how she expected the model to illustrate a process associated with solving the problem. Her rationale indicates that she views models not simply as a static representation used to present the final answer but as a tool to solve problems. Anticipating that the model would capture a solution process, she appeared to associate it with subtraction, as the only process visible to her was takeaway. While Carlota did not see $\frac{5}{12}$ in the model, it was not because she lacked a quotient understanding of fractions. She simply did not connect such an interpretation to the model given, as it was removed from the action presented by the context. In her justification for why none of the answer choices were correct, she provided detailed solutions to both a) and b), including her own process-oriented representations similar to Figure 3.

These three students, as illustrated by Carlota, expected a correct model to represent the equal sharing process outlined by each of the possible answer choices. She possessed a strong understanding of the context, could draw a model that captured the equal sharing process, and was able to use it to arrive at a correct solution. Her reasoning was simply not rewarded because the model provided did not embody this action.

Category 3: Dual Simultaneous Interpretations-Process and Product

The last three students, Eric, Bojan, and Josh, initially interpreted the model similar to students in the first category but were able to build on this initial understanding. They all imposed the process of equal sharing on the model and then reconceptualized it as the product of this process. As we illustrate below, because the static nature of the model required them to abstractly enact this process on the model, they arrived at two different, yet logical solutions.

Eric believed the correct answer was a). When asked to clarify his thinking, he first explained that each square represented one of the 12 individual jars and that the 5 shaded rectangles represented the gallons of tea. He then began to model the distribution process, envisioning the 5 gallons of tea being poured into the 12 jars. To do so, he created his own diagram of a single gallon and divided it into 12 parts. He then imagined pouring $\frac{1}{12}$ of each of the 5 gallons into this newly drawn gallon, explaining, "When we take 5 and put it into the 12 we are splitting the 5 each into 12. Each whole is getting split into 12. That first jar is going to get $\frac{1}{12}$ of one of the gallons then we can multiply it by 5 to get the 5 gallons. Which when we multiply we get $\frac{5}{12}$." With this explanation, he clarifies that the 12 rectangles no longer represent the individual jars but rather now collectively represent a single gallon.

Eric's solution is notable because he was able to impose his understanding of the distribution process on the given model. In addition, he was able to reconceptualize the model and construe it as the product of this process. To see both the process and product in the model required him to redefine the meaning of the different components in the model and interpret them simultaneously in two different ways. To facilitate this transition, he created his own intermediate diagram of the distribution process but then reconnected it to the original model.

Bojan and Josh engaged in similar reasoning as Eric but applied their solution method to answer choice b) and c) respectively, resulting in valid, but completely different interpretations of the model. Although Bojan and Josh selected different problems, their interpretations of the model were similar. To illustrate this thinking, we look at Josh's explanation of c). He began by explaining how he imagined the 12 squares representing the 12 L of water that needed to be split evenly among the 5 jars, represented by the 5 squares that were shaded. He then started to abstractly pour a liter of water into each of the 5 jars until he stopped at the remaining 2 liters. At this point he had filled each of the 5 beakers with 2 liters of water, a total of 10. He then started to separate the final 2 liters of water into groups of 5, giving 1/5 from each liter to the 5 beakers. In the end, the 12 liters had been equally distributed and each of the beakers was full of 2 and 2/5 liters of water, a result he fully clarified.

For Josh (and similarly Bojan), he imagined $2\frac{2}{5}$ inside each of those shaded beakers, the same way that Eric saw the model as one jar with $\frac{5}{12}$ of a gallon. The ability of both students to impose their own thinking on the model demonstrated a rich understanding of a process interpretation of the model as well as one connected to the numerical result.

Conclusion

While standardized tests have begun to broaden the representations used to assess students' understanding of mathematical concepts, this study demonstrates that test makers must consider more comprehensively the type of thinking students will engage in when presented such representations. While on the surface the inclusion of more concrete visual models appears to be a productive characteristic, these results highlight the pitfalls of creating a standardized test with the expectation of a single interpretation of an abstract representation. The design of this problem seemed to encourage a *pseudostructural* conceptualization of the representation as the model provided was divorced from the equal sharing process. Such a narrow representation proved highly problematic, resulting in confusing the vast majority of students we interviewed with very few of them able to demonstrate their understanding.

Models, like all representations, do not have a singular meaning. Without proper consideration, students can and will express rich understandings very different from the answer deemed correct. One lens when designing test questions to consider is the degree to which the model allows students to conceptualize both the product and process. Students who can see both a process and a product within a representation possess a deeper understanding of the underlying mathematical concepts. However, such rich understanding can be marginalized due to high stakes testing.

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THE IMPACT OF PROBLEM-SOLVING DISCUSSIONS ON HEURISTIC USE AND METACOGNITION

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The purpose of this mixed-methods study was to investigate the impact of daily problem-solving discussions on fifth-grade students' use of heuristics and metacognitive awareness when problem-solving. The sample consisted of 74 students taught by five distinct teachers from three schools in a Mid-Atlantic State. The results suggest that problem-solving discussions impact students' use of heuristics in class, but that these outcomes may not transfer to test-taking environments. No significant impact on students' metacognitive awareness was found. These results add to the literature base on how to teach problem-solving in elementary schools.

Background

Teaching through problem-solving has been shown to increase students' creativity, motivation, transfer, mathematical understandings, and the prevalence of positive mathematical dispositions (Boaler, 2002). Consequently, problem-solving has been identified as a key outcome of mathematics courses and as a conduit for deeper learning (Cai, 2003; Lester, 2013). Past studies have highlighted factors undergirding problem-solving such as metacognition (Flavell, 1979), large repertoires of heuristics that can be flexibly applied to problems (Jitendra et al., 2015; Koichu et al., 2007), content knowledge, and beliefs (e.g., Schoenfeld, 2013). However, little is known about how best to utilize these relationships in teaching (Lester & Cai, 2016).

Promisingly, the positive impacts of problem-solving may be enhanced when students are given opportunities to problem-solve in discourse-rich environments. Such environments allow students to explore their thinking as they justify their claims (Cai, 2003; Lester & Cai, 2016) and reflect on and regulate their thinking (e.g., Lester, 2013; Rosenzweig et al., 2011). Evidencing this, Koichu et al. (2007) used think-alouds to conclude that teaching heuristics through discourse significantly increased mathematics achievement for students. Thus, coupling heuristic instruction with metacognitive supports may be crucial in developing problem-solving instruction that allows students to flexibly apply their learning (Jitendra et al., 2015).

Despite the links between discourse and mathematics proficiency (e.g., Goos et al., 2002; Koichu et al., 2007), few studies have considered whether daily problem-solving discussions impact students' problem-solving proficiency, metacognitive awareness, and heuristics use when problem-solving. Thus, the purpose of this study was to explore the impacts of integrating problem-solving and daily discussion-based routines. Framing this, the research questions were, when compared to a control group, does the daily use of problem-solving discussions:

1. impact fifth-grade students' problem-solving proficiency?
2. impact fifth-grade students' use of heuristics when problem-solving?
3. increase students' metacognitive awareness?

Theoretical Framework

The theoretical lens of Social Constructivism (Fox, 2001) framed the current study. Specifically, the study was designed and interpreted at the intersection of assumptions that knowledge is a product of social interactions and that knowledge is constructed through the lens of individual experiences (Fox, 2001). Consequently, the intervention described below included components that allowed for students to pose and interpret problems in light of personal experiences, but then to also share personal thinking and consider the thinking of others through classroom discussion. This framing is supported by the aforementioned research that suggest that discourse is a crucial component in problem-solving, in developing students' abilities to problem-solve, and in increasing students' effective use of heuristics (e.g., Goos et al., 2002).

Description of Intervention

As noted above, using problem-solving in discourse rich environments that support metacognition has shown promise (e.g., Koichu et al., 2007; Rosenzweig et al., 2011). Coupling this with the social constructivist framing, the intervention consisted of "problem-solving discussions," which were operationalized as five to fifteen minute full-class discussions around a problem. These discussions generally consisted of a two to four-day sequence in which the students first activated and considered their own experiences and curiosities by posing mathematical questions around a given prompt. Students then unpacked a number-less version of the problem, shared different heuristics that they believed could be used to solve the problem that included quantities, and finally solved the problem and discussed different solutions.

Exemplifying this process, the following basic problem was used to introduce the process to students prior to using more rigorous problems: First, a prompt of "Hunter is riding his bike this weekend" allowed students to pose mathematical problems (e.g., "how fast did Hunter ride?"). This was then followed up on the second day by "Hunter is riding his bike in a challenge this weekend. How far did he ride on Sunday?" In considering this prompt, students asked questions and then evaluated these questions as a group based on their likely usefulness in solving the problem (e.g., "what is the total distance that Hunter rode?" was deemed to be potentially useful while "was Hunter wearing a helmet?" was deemed to be likely irrelevant to the given problem). Finally, on the third and fourth days, students solved the problem: "Hunter is riding his bike this weekend. He wants to ride 5 miles in under 48 hours. On Saturday, he rode $3\frac{1}{5}$ miles. How far did he ride on Sunday?" Within this, teachers were expected to do the following:

1. Choose problems that allowed for productive struggle
2. Provide opportunities for students to engage in sense-making
3. Allow students to develop and share their own models of thinking and heuristics
4. Focus conversations on both problem-solving heuristics and content knowledge
5. Record student thinking in a visible location and ask questions to encourage reflection and metacognitive awareness

In these ways, the problem discussions were designed to elicit student thinking and facilitate discourse and reflection around various aspects of problem-solving and the use of heuristics.

Methodology

Participants

The participants consisted of 74 students that were drawn from six, fifth grade math classes. These classes were taught by five distinct teachers from three different elementary schools within a single school district in the Mid-Atlantic region. The students were divided into a control group that included 38 students (17 male and 21 female) that were taught by two distinct teachers, and an intervention group containing 36 students (18 male and 18 female) who were taught by three distinct teachers. Two students from each group were missing pre-test data and were excluded from the quantitative analysis. Prior to the intervention, all three teachers from the intervention group and one from the control group reported that their schools—starting as early as second grade—required that students use a single heuristic (model drawing) on all classroom assignments and assessments. Contrastingly, the teacher of the other two control-group classes, who taught 31 of the 38 students in the control group, stated that they teach problem-solving by using "different strategies, manipulatives, and drawing to [help students] understand a problem."

Data Collection

In this mixed-methods study, participating students were pre- and post-tested on problem-solving and metacognitive awareness. The problem-solving measure was used to address the first two research questions related to problem-solving proficiency and heuristic use. The measure was developed using nine questions drawn from released state tests that had been well validated by the state. However, five questions on the post-test had to be modified due to potential compromises and were validated by mathematics and assessment experts to ensure consistency in structure, content, and rigor. Meanwhile, the Jr. MAI (Sperling et al., 2002) was used, with permission, to address the third research question related to metacognitive awareness.

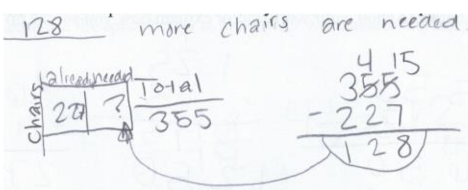
After pre-testing, intervention teachers attended 12 hours of training on problem-solving, metacognitive questioning, and on facilitating meaningful classroom discourse before facilitating problem-solving discussions for nine weeks. However, it is worth noting that actual implementation ranged from 23 days to 36 days for the three intervention teachers out of the 45 possible school days. Finally, semi-structured interviews lasting approximately 30-minutes were conducted with the intervention teachers—Lenny, Olivia, and Bailey. These interviews were designed to understand the fidelity of implementation and to collect qualitative data on the teachers' perspectives in order to better understand and interpret the quantitative findings.

Data Analysis

Problem-solving pre- and post-tests items were scored as correct or incorrect and assigned a value of "1" or "0," respectively. Points were then summed to provide a score out of nine. An ANCOVA was used to analyze group differences while controlling for pre-test scores. Student problem-solving measures were then coded based on the heuristic that the student used to solve each problem. This process resulted in the development of seventeen unique codes, the four most common of which are shown below in Table 1. For example, Figure 1 shows a student work sample that was coded as "used a model drawing and then applied a common algorithm."

Figure 1

A Student Work Sample Showing a Model Drawing



Jr. MAI pre- and post-tests were scored by converting Likert scale responses of "Never", "Sometimes", and "Always" to a 1, 2, or 3, respectively. These scores were then averaged across the 12 questions and analyzed using an ANCOVA to control for pre-test scores. Finally, initial codes were generated from transcribed teacher interviews using line-by-line coding, which were then grouped to form themes (Charmaz, 2014). For example, the quote "towards the end of the study, some of them came up with some really good strategies" was coded as "student strategies improved over the study," and supported the development of the theme "many students generated more problem-solving strategies over the course of the study."

Results

Qualitative Analyses

Qualitative analyses of interview data through the lens of social constructivism resulted in seven themes. Selected quotes evidencing the two themes relevant here are included below.

***Theme 1: Students Began to Generate More Problem-solving Heuristics and Ask More Questions*.** Evidencing the first theme, Olivia, noted that "at the very beginning [of the study]…[the students] were trying to draw all these elaborate flags that didn't match the problems," but that "towards the end of the study, some of them came up with some really good strategies…[like] drawing pictures to model their thinking instead of doing model drawing necessarily." Moreover, Bailey said that, as a result of the discussions, students "started to internalize and question, do I need that information? Do I need to know more information? …What piece did I need to know? What piece didn't I need to know?" Finally, Lenny said that "when a student engages their prior knowledge and experience to develop a strategy, they construct a closer connection to the problem," thereby positing that classroom discussions allowed students to draw on their personal experiences to construct heuristics.

***Theme 2: Students Began to Value Their Thinking and the Thinking of Their Peers*.** Evidencing the second theme, Olivia noted that "because of the different answers [the students were] more likely to share their answers and explain their thinking." Similarly, Bailey said that, by the end of the intervention, students were "more open to hearing from a different perspective… [and] looking at the question in a different way" at the end of the study. Finally, Lenny reflected that students transitioned from a "how am I supposed to solve [the problem]?" mentality to a focus on understanding. Lenny attributed this change to the focus on student discourse which allowed students to take more "initiative…[and] led to… [heuristics being] developed by students… [rather than by] teachers."

Overall, all three teachers believed that students' use of heuristics, abilities to ask meaningful questions, and consideration of their own thinking as well as the thinking of their peers, improved as a result of the study. Consequently, the teachers believed that the diversity of thought caused students to utilize diverse heuristics when problem-solving.

Quantitative Analyses

Analysis of the problem-solving pre- and post-tests showed that both groups improved on the post-test with the mean scores increasing from $\bar{x} = 5.912$ ($SD = .327$) to $\bar{x} = 6.472$ ($SD = .286$) for the control group, and from $\bar{x} = 6.324$ ($SD = .363$) to $\bar{x} = 6.971$ ($SD = .303$) for the intervention group. However, group differences were not significant when controlling for pre-test scores, as analyzed using an ANCOVA ($F = .719$, $df = 1$, $p > .05$). Similarly, analysis of student work showed minimal changes in the heuristic's students used when problem-solving. As evidenced by the four most prevalent codes shown in Table 1, students tended to rely on learned algorithms. The only notable difference between groups was that the intervention group also relied heavily on model drawings—a difference that is likely attributable to prior instruction.

Table 1

Frequencies of the Five Most Prevalent Heuristics used when Problem-solving

	Pre-test (Control)		Pre-test (Intervention)		Post-test (Control)		Post-Test (Intervention)	
	N	%[a]	N	%[a]	N	%[a]	N	%[a]
Used a common algorithm to add, subtract, multiply, or divide.	196	60.5	126	41.2	201	62.0	138	45.1
Used a common algorithm to add or subtract fractions.	48	14.8	56	18.3	49	15.1	43	14.1
Used a common algorithm to find a common denominator for fractions.	27	8.3	17	5.6	12	3.7	14	4.6
Used a model drawing and then applied a common algorithm.	0	0	67	21.9	0	0	63	20.6

[a]Percentage calculations are frequencies compared to the total heuristics for each measure.

Finally, pre- and post-test scores for the Jr. MAI remained stable for both groups when controlling for pre-test scores using an ANCOVA ($F = 2.085$, $df = 1$, $p > .05$), with mean scores decreasing from $\bar{x} = 2.276$ ($SD = .213$) to $\bar{x} = 2.213$ ($SD = .247$) for the control group, and increasing from $\bar{x} = 2.268$ ($SD = .202$) to $\bar{x} = 2.286$ ($SD = .250$) for the intervention group.

Discussion

The contrasting results of this study are difficult to interpret. Teacher interviews suggest that the discourse-rich environment caused students to use a wider array of heuristics when problem-solving, began to reflect on their own thinking, and considered the thinking of their peers. These findings align with prior research (e.g., Jitendra et al., 2015; Koichu et al., 2007) that suggests that metacognition and the use of heuristics are intrinsically linked when problem-solving, and that student-centered discourse is a promising avenue for developing these skills. Contradicting this, quantitative analyses do not suggest that meaningful growth occurred in terms of students' abilities to problem-solve, use diverse heuristics, or maintain an awareness of their thinking.

One potential explanation is that the intervention did impact students' abilities to utilize heuristics when problem-solving—as noted by the teachers—but that these learnings did not transfer to the post-test. Given that, prior to the intervention, most intervention students had been required to use model drawing on all tests and were penalized when they did not, it is possible that the formal testing environment of the post-test activated these experiences and may have inhibited students' willingness to utilize learned heuristics. Supporting this possibility, Olivia reflected that many of her students had a hard time initially transitioning away from only using model drawing, and suggested that the intervention should start "in second grade." Thus, given that identities become less malleable over time (e.g., Langer-Osuna & Esmonde, 2017), a longer intervention may have been necessary to overcome this barrier.

Research also suggests that reflection is vital in the development of more robust cognitive and metacognitive processes (e.g., Zelazo et al., 2018). Considering this in terms of problem-solving, context situated reflection of heuristic use supports students in building conceptual frameworks that may allow for better understanding and transfer of problem-solving skills (e.g., Hamilton et al., 2007). Although informal reflection was a component of the discussion designs, it is likely that maximizing the impact of any problem-solving intervention requires incorporating structured reflection (e.g., through writing or reflection tools) into the intervention.

Overall, using daily, discourse-based problem-solving routines demonstrate potential in supporting students in heuristic literacy and metacognition. Despite this, more research is needed to analyze the impact that discourse-based problem-solving has on students' use of heuristics and metacognitive awareness when problem-solving. Specifically, future researchers should consider iterations of the intervention that increase the duration of the intervention that include a more explicit focus on transfer and on reflection.

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Reestablishing Connections for Post-Secondary Students

CHOOSING A RESEARCH QUESTION IN APPLIED MATHEMATICS, FROM MENTORS TO NOVICES

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Based on observations made during the project proposal week of an undergraduate research program in applied mathematics, this paper explores the role of faculty in guiding students in developing a research question and an accompanying model. Results suggest that students should be pushed to become experts in the background subject matter, while mentors take the lead mathematically. Key skills for developing a research question include: defining the temporal/spatial focus, exploring broader impacts of the work, and anticipating possible mathematical results to help define the question. Constant dialogue on both sides about the scientific mechanisms informing mathematical choices was critical to model development.

Mathematics as it is taught in secondary and post-secondary classes differs greatly from mathematics as it is practiced by professionals and from the needs of partner disciplines (Ganter & Barker, 2004; Ganter & Haver, 2011; Lewis & Powell, 2017). Reports from industry and professional societies repeatedly emphasize the importance of professional skills in communication, collaboration, problem solving, mathematical modeling, and creativity on top of a solid foundation of procedural skills and coherent mathematical understandings (Bliss et. al., 2016; Ganter & Barker, 2004; Ganter & Haver, 2011). These soft-skills do not come free with fluency in mathematical skills or sophisticated mathematical understandings.

The skill of interest to this report is described by Bliss et. al. (2016) as "Distilling a large ill-defined problem into a tractable question," (p. 72) which I will call "developing a research question." Smith et al. (1997) found that graduate students in mathematical biology struggle with developing research questions that are both biologically interesting and mathematically tractable.

Mathematical modeling education research has little to say on the development of this skill. Research in this area typically focuses on students working pre-chosen tasks (Bliss et. al., 2006; Gravemijer, 1994; Lesh & Doerr, 2003). Sometimes these tasks are quite open, and students go through cycles of model development; however, in assigning a task to students, there are constraints placed on students as part of intentionally guiding the students' conceptual development (ibid). It has been argued in the past that these constraints limit students' experiences in developing their own research questions (Castillo-Garsow, 2014; Castillo-Garsow & Castillo-Chavez, 2015).

Camacho et al. (2003) found that choosing one's own project and research question creates situations in which students take the lead in researching topics far outside a mentors' area of expertise, essentially reducing the mentor to a role of consultant rather than leader. These unique situations create opportunities to identify key components of soft-skills such as developing a research question. For example, in studying student-chosen projects, Smith et al. (1997) found that students' and mentors' goals for modeling, as well as their beliefs about the value and purposes of modeling, impacted their ability to develop a research question that resulted in a scientifically relevant and tractable model.

This study follows a single group of undergraduate students in the process of developing a topic of their own interest into a research question and accompanying mathematical model. Because the students in this project worked in close and constant collaboration with both graduate and undergraduate mentors, we can see how mathematicians at different stages of their career view the task of developing a research question and accompanying model.

The purpose of this study is two-fold: (a) to begin the process of identifying specific goals, skills, and values that are critical to developing a research question and model in applied mathematics and (b) to provide guidance to mentors of student-led applied mathematics projects by identifying effective, transferable interventions.

Methods

This study occurred in the fifth week of an eight-week summer REU in mathematical biology. Prior to this study, the students had taken a three-and-a-half-week course consisting of lecture, computer lab work, and textbook exercises in dynamical systems. Following this course work, students self-recruited into groups of three to five, and chose a topic of interest. During the fifth week, students made daily presentations on their topic to a panel of faculty and graduate mentors who provided feedback. In the final three weeks of the program, students completed the analysis of their model and wrote a technical report on their project. Four groups of students chose to participate in the study, and these preliminary results are from the analysis of the first group. This group was chosen to analyze first because the project was judged by participating mentors to be the closest to a typical project in the REU, and because the success of the project could be determined by publication in a prominent journal. The citation for the publication is omitted to protect the privacy of the participants.

The group of students in this study was formed of five undergraduate students who chose to construct a model for controlling a disease that is transmitted between multiple species of animals. They made six presentations during proposal week to a panel of faculty and graduate mentors. One presentation (five) was cut short by the mentors who did not believe that their feedback was necessary at that time. Thirteen of these mentors made comments on the students' work. Each proposal conference was video and audio recorded, and the audio recordings were transcribed. These transcripts, along with the technical report written at the end of the summer program were taken as data for analysis. The published version of the students' project, which resulted from unrecorded collaboration after the program ended, was omitted from this study.

Transcripts were initially open-coded and then axially coded (Strauss & Corbin, 1990) using qualitative data analysis software. This coding provided a visualization for how the content of the presentations and the priorities of the participants changed as the project developed. However, this initial coding did not give a sense of the impact that mentors' comments had in influencing the direction of the project, so a second analysis was performed. In this second analysis, mentor comments were isolated from the transcript, and each mentor comment was coded individually for its content, and then in the context of the transcript and the final paper with an eye to how the content of that comment was revisited in students' future work. Over the six daily presentations, 269 comments from 13 mentors were isolated, coded, and analyzed.

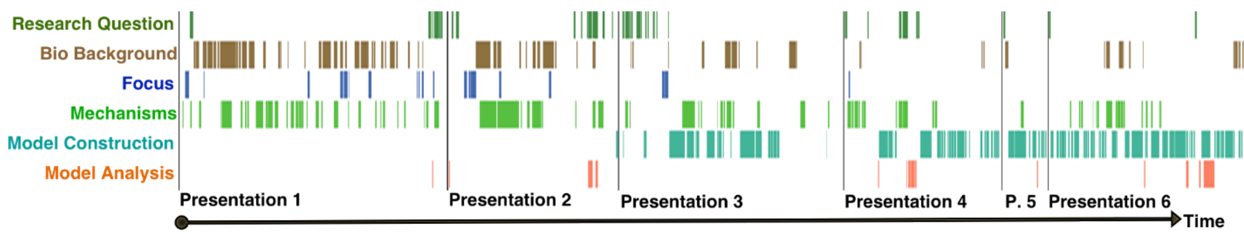
Results

Initial Coding

The initial coding provided a sense of the structure of the presentations. Figure 1 shows the top level codes for the topics being discussed by both students and mentors during the presentation. Topics generally fell into nine broad categories in this analysis, only six of which are discussed in this paper due to space limitations. The first code was related to discussions of the students' research question, such as asking students to present their research question or discussing what makes a good research question (ex: "I'm curious what a good measure is for whether [your intervention is] effective"). Background codes referred to discussion of the biological background situation in general, but not specific to developing the model, such as the initial literature review into the behavior of the disease or the life cycle of an animal (ex: "But if you put those same surface proteins on the bacteria that they do produce an immune reaction to, then they'll build up antibodies with that initial infection, they will start fighting off the bacteria”). Focus codes referred to codes that were about defining the problem space to a specific geographic, demographic, spatial, or temporal region (ex: “the other data sets were from Indiana, which is a very different ecological setup than in this area”). Mechanism codes referred to key processes that informed model development, such as identifying stages that individuals pass through, choosing variables, describing the precise way that individuals counted by those variables interacted, and proposing specific intervention strategies (ex: “Yes, so every time they are moving onto the next stage… that's when they can pick up the bacteria”). Model codes referred to the development of the equations and corresponding flow diagrams to be used (ex: “it should be NI + NS. Infected and susceptible”). Lastly, model analysis codes referred to mentors anticipating the results that students might get (ex: “Regardless of whether these treatments are able to reduce R0 below one, they will reduce the endemic prevalence somewhat”).

Figure 1

Presence or Absence of a Code in the Transcript Over Time



Note: Timeline of topics discussed by both mentors and students during the students’ six presentations. Color indicates the topic is being discussed at that time, while white indicates the topic is not being discussed at that time. Black vertical lines separate individual presentations.

Three topics were prominently discussed in every session (Figure 1): the research question (dark green), the background biology (brown), and the specific mechanisms that would inform model construction (light green). Two days of presentations were devoted to these topics before the research question is defined and model construction began, and during model construction, these topics – particularly mechanisms – continued to be revisited.

Much of the early discussion in defining a research question focused around defining the scope of the project (Figure 1, dark blue). Prior to presentation three, a prominent feature of the discussion was placing specific bounds on the scale of the study: the geographic location to be modeled, the specific populations to be studied, and time scale to be used. These questions provided direct guidance in refining the research question.

An unanticipated result is that the discussion of model analysis (Figure 1, bright orange) preceded the development of the model (cyan), and even the final determination of the research question itself (dark green). Discussion of possible methods of analysis created a common language that the mentors used to discuss the research question in an unfamiliar topic. By discussing potential mathematical results, the mentors helped guide the research question to being one that could be defined mathematically in the form of a model.

The Second Coding

The second coding isolated mentor comments and related the impact of each mentor's comment on the project to the coded content of the comment. Mentor comments were coded by content, and then the impact of each mentor's comment was coded in three ways (Duration, Fidelity, and Direction). Duration was coded as Final, Local, or None depending on if students responded to comments in the final paper, in presentations, or not at all. Fidelity was coded as Pivotal, Direct, or Tangential, depending on whether students based a key aspect of the project on the mentor comment, followed feedback faithfully, or made changes that were merely related to the feedback. Direction coded the novelty of the mentor's suggestion itself as Identical, Similar, Distinct, or Novel, depending on if the comment was something students had discussed before in presentation, related to an idea students had discussed in presentation, different from something students had discussed in presentation, or a new idea that students had never discussed in presentation. Coding was based only on the evidence available within the presentations and the final paper. It is possible that students may have thought of an idea coded Novel and never presented it, but these possibilities were not a factor in coding.

For examples of codes, the mentor comment "One box can cover how big an area?" was given a code of Similar because students were planning a spatial model, Direct because students followed up and found an answer, and Local because students did not include space as a factor in their final model. The mentor comment "How much are these boxes?" was coded as Novel because students had not proposed to look at cost prior to this question, and Final because cost-optimization played a role in the final paper. This comment was coded as Direct rather than Pivotal because students initially only answered the question. The following day the comment "you could impose a cost structure on it" was coded as Pivotal, as students changed their research question after this second comment, but Similar because students had already discussed cost. For an example of Tangential, several mentors that students investigate the feeding behavior of the animal, and students did look for papers on this topic, but never incorporated the idea into their project. Comments that most reliably impacted the final paper were questions that asked students to define the research question or the focus of the project more precisely, and comments that assisted students with model development or choosing a methodology. Comments that received no response from students were ones that requested changes to the direction of the research question or the focus, or asked students to make or explain decisions about model development.

A look at the relationship between Direction and Fidelity showed a similar story. Comments that helped students do what they were already going to do (Identical or Similar Direction) were followed faithfully (Direct Fidelity) 78% and 67% of the time (Identical and Similar respectively). Comments that asked students to make drastic changes (Distinct Direction) were unpredictable, resulting in 43% Pivotal Fidelity, 33% Direct Fidelity, and 25% Tangential Fidelity (after rounding). Comments that introduced new ideas (Novel Direction) were either comments that resulted in radical changes to the project (33% Pivotal Fidelity), or were followed only in a perfunctory way (50% Tangential Fidelity).

Mentor comments that students took as pivotal advice were mentors explaining mechanisms, asking students to explain mechanisms, or mentor suggestions for methodology or analysis. Comments students followed directly were asking students to explain model development decisions, or suggesting specific changes to the model. Students compromised (Tangential Fidelity) on requests to do additional background research or find more data.

Comments that developed ideas students haven’t thought of (Novel or Distinct Direction) included suggesting or providing additional background, asking students to explain mechanisms, or suggesting the impact of their research (how it might affect others). Comments that drove students deeper into existing directions (Similar or Identical Direction) included asking students to decide on a model or analysis, or asking students about the impact of their research.

Discussion

In broad strokes, mentors focused on asking questions about the background situation (which they were less knowledgeable of), and provided direct suggestions about the mathematical construction of the model (which was within their area of expertise). The timeline of the presentations also followed a trajectory of background to research question to model. However, a deeper dive into the data identified skills and mentor actions that were not immediately obvious.

The data analysis identified specific ways in which mentors guided students in the development of a research question: (a) mentors pressed students to precisely define the spatial, temporal, and demographic focus of the study, (b) mentors suggested possible impacts or implications for the research, and (c) mentors anticipated possible mathematical analysis or results, which helped guide the choice of research question.

Students appreciated feeling in control of their project. Mentor comments that helped students to do what they already planned to do were followed much more faithfully. Mentor comments that requested large changes to the project were sometimes of pivotal importance, but more frequently were ignored, or partially followed. The difference was the background expertise of students and the mathematical expertise of mentors. Students were reluctant to make decisions about the mathematical construction of the model, but welcomed the opportunity to direct their own project and be experts in the background subject matter.

Lastly, the discussion of mechanisms played a key role in every aspect of research question and model development. Mentors directly pressed students for deeper scientific understanding until it could be made mathematically precise. This skill of examining how the background process worked, and choosing an expression to describe that process was critical for model development and was exercised consistently throughout all the presentations.

A limitation of the study is that of the 269 comments made by mentors, slightly over half (141 comments) received no response from students. Part of the reason for lack of response of students is the large number of mentors who frequently talked over each other or interrupted, giving the students insufficient time to respond. It is impossible to determine how students might have responded to these comments that were interrupted.

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STUDENT ATTITUDES IN A REAL-WORLD INSPIRED SECOND STATISTICS COURSE

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A second algebra-based college-level statistics course was redesigned with the desire to help students make connections between class content and its use in the real-world. Attitude changes were investigated using the SATS-36. Positive changes were found for Cognitive Competence, Affect, and Difficulty while no significant changes were found for Interest or Value.

Background and Literature Review

Investigating students' attitudes toward statistics has been the focus in many research studies. Gal and Ginsberg (1994) were one of the first to emphasize the need to assess student attitudes as they are likely linked to the students' perceived difficulty of the subject. Huynh et al. (2014) discovered that students found mathematics and statistics unenjoyable to learn so they wanted to figure out what could be done to improve students' attitudes towards the subject.

Many instructors have implemented real-world applications and even real data as recommended by the GAISE College Report (American Statistical Association, 2016) hoping to improve students' connections and their attitudes toward the content. The efforts at helping students make these connections has been done in a variety of ways. Some researchers have tried to make this link by having students collect their own data (Carnell, 2008, Huynh et al., 2014). Other statistics instructors have incorporated activities into their course with the hopes of improving students' attitudes towards statistics. Paul and Cunnington (2016) investigated implementing the GAISE recommendations in an introductory statistics course. However, these studies often found either no significant change in attitudes or negative changes.

There have been many studies that have investigated how students' attitudes toward statistics change in an introductory college course; however, what about the students who must take a second course housed in a statistics department? Are we able to improve their attitudes by designing the course with their future applications in mind? Knowing that students' attitudes tend to become more negative during the first course (Schau & Emmioğlu, 2012) and that they will likely carry these attitudes into the second statistics course, it was of interest to design the second course in a way that would allow students to see the applicability of what they were learning to their futures. More specifically, the course was to be designed in a way that students could see how statistical techniques could be used with real data and could also learn how to communicate this information in a non-technical manner. It was then of interest to measure the attitude changes over this second statistics course to see if there were more favorable outcomes.

Second Statistics Course Topics

This second algebra-based college-level statistics course is aimed at a business major audience, which does not include mathematics or statistics majors. Topics included in this course include two-sample means, ANOVA, non-parametric tests, two proportions, chi-square test for independence, regression techniques (simple linear, multiple linear, quadratic, and indicator variables), and time-series. The overarching focus for this course is to determine the most appropriate analysis method and provide thorough interpretations. All p-values and confidence intervals (except for two proportions) are generated using software; students should already have a solid foundation of the computations from their first course. However, students are still expected to do other by-hand calculations such as various test statistics (e.g., chi-squared, non-parametric test statistics), predictions for regression, and finally, be able to fill-in missing pieces of outputs (such as test statistics, expected counts, and p-values (from multiple choice options)).

Course Design

Although the content of the course remained the same, how the class was taught, as well as the assessments that were given, were designed to allow for students to see the applications of the course content both immediately as well as to their future careers.

One aspect that was created in this new design was the inclusion of "Laptop Days" (LDs) throughout the course. During these LDs, students worked in pairs during the class period and were asked to have at least one laptop between them. Students were given a real dataset (or simulated dataset based on a real result) that pertained to the topic at hand. Each pair was given an assignment where they were provided a description of the dataset and goals of the study. For classroom purposes, students were also provided with questions that were similar to what would show up on a quiz or test. Students were asked to work together to analyze the data in a way that answers the goals that were given. Although students were provided with questions that would be similar to those found on a quiz or exam, that is not what they were asked to turn in; instead, they were asked to turn in a written report (not a research paper) of their findings.

After completing the analysis for their LD, students worked on writing a report where the intended audience was their (future) boss. Students were asked to provide goals, variables recorded, a descriptive summary of the data, inferential results, and recommendations. In addition, students were asked to include an appendix that contains the output used and why they chose the corresponding analysis method. Reports were typically about two pages in length (excluding the appendices) and were due the class period following the LD. This report was intentionally designed to not be a formal research paper, but instead, be an overview of the important and interesting findings. The idea was for students to get used to briefly and clearly communicating important results in a non-technical way. In addition, these reports served as mock write-ups for the business world.

Besides LDs, the typical class day was a mix of lecture and discussion. This included both writing on the board and software demonstration. Having a desire for students to think beyond the last step of a hypothesis test or confidence interval, the question "now what?" was asked and discussed quite often. Why is this result important? What should happen next? As an example of this discussion, suppose a company was testing two new sodas for market. It was found that soda 2 was preferred to soda 1. Although students wanted to jump to the conclusion that soda 2 should be marketed, they were led to think about if the customers even liked soda 2 or did they dislike soda 1? They then realized that they had not actually answered these questions and began to see the importance of follow-up analyses.

Students' LD grades accounted for 10% of their course grade. The rest of the grade was broken down into homework (10%), in-class quizzes (10%), three mid-term exams (45%), and a final exam (25%). Each of the exams included an in-class and take-home component, where the take-home component accounted for 20% of the exam score and involved using software.

Research Questions

After intentionally designing the course to improve student connections, it was of interest to see how students' attitudes changed during this second statistics course, if at all. This led to the following research questions that will guide the analyses:

- How do attitudes change over the duration of a second required statistics course? (RQ1)
- Is the instructor of the first course a factor when measuring attitude changes in the second course? (RQ2)

Participants

The university this study took ace at is a regional university with a large proportion of commuter students and an undergraduate enrollment of around 12,000 students. The course being studied is a business statistics II course, where students must have successfully completed a first semester statistics course with a C- or better. All students enrolled in the course did so because it was required for their major. This study included 96 students with a mean age of 21.3 years (3.2 year standard deviation) and 52.08% identified as female. Some previous literature has shown that the mean age of students in a first semester statistics course tends to be around 21 years of age and 58% identified as female (Wroughton et al., 2013). Thus, one can note that there is about the expected difference in mean age of the students (as a semester is roughly 0.5 a year). In addition, there is not much difference in the percentage of students who were female.

There has been some research on the substantial impact that an instructor can make on both student performance and their attitudes (Xu et al., 2020). Thus, in order to control the instructor effect as much as possible, all students used in this study had the same instructor for the second course. However, knowing that students' first statistics instructor could have an impact on their feelings about the second course, the students' first statistics instructor (author or not author) was also investigated when looking at attitude results.

Instrument: Survey of Attitudes Towards Statistics (SATS)

The Survey of Attitudes Towards Statistics (SATS) (Schau, 2003) has been a widely used tool for assessing students' attitudes in an introductory statistics course. The SATS-36 instrument consists of 36 7-point Likert scale questions that are broken into six components (Schau, 2003): Affect, Cognitive Competence, Value, Difficulty (note that a higher score here actually means the student finds the class *less* difficult), Interest, and Effort.

The SATS-36 is intended to be given as a pre- and post-assessment at the beginning and end of the semester. Schau and Emmioğlu performed a large-scale study across multiple years and many universities. They reported the SATS results of this first semester statistics course in their 2012 paper (p. 91). They found that the mean change in scores (post-test-pre-test) for Affect, Cognitive Competence, and Difficulty were slightly positive (0.1 to 0.15 point increases) while there were negative changes in Value, Effort, and Interest (-0.5 to -0.32 points). There is some concern over the Effort component as the mean is generally quite high for students on the pre-test leaving little to no room for increases. As Schau and Emmioğlu state, the best statistical methods to deal with this type of issue are not yet clear. In addition, Schau and Emmioğlu indicate that a difference of 0.5-point or more is considered to be an important finding. There were two components in their results that are at this magnitude: the negative changes in Value and Interest. These results found by Schau and Emmioğlu (2012), were treated as the historical comparison for the completion of a first statistics course.

The pre and post-test of the SATS-36 will be used to measure attitudes of the students in the second statistics course. In all situations, the pre-test was given after the first day of class during the first week and the post-test was given the week prior to final exams, both of which were administered online.

Methods & Results

In order to first investigate attitudes, the SATS-36's components' mean changes will be of interest. Although all six attitude components were recorded, Effort will not be investigated due to its historical problems with being extremely high on the pre-test leaving little room for an improvement in student attitude. As a 0.5-point change in mean attitude scores is considered an important finding, and with sample sizes ranging between 23 and 96 over the duration of the analysis, when performing all tests of significance an overall 10% significance level will be used. These sample sizes correspond to powers ranging between 63% and 99%, with 90% power occurring at a sample size of 44. When multiple tests are performed, a Bonferroni adjustment will be made.

The first part of the analysis will be guided by RQ1, examining just the five attitude components (Affect, Cognitive Competence, Value, Difficulty, and Interest) of the second statistics course. Investigation of the QQ plots of the differences and insignificant tests of normality (smallest p-value = 0.222) allow for multiple paired t-tests (with a Bonferroni adjustment) to be performed. As previously mentioned, the SATS-36 was administered as a pre- and post-test for the course. Only students who completed both the pre- and post-test are included in this analysis. Results for the 96 students are found in Table 1 below:

Table 1

Student Attitude Survey Results of Second Statistics Course – Mean Scores

Component	Pretest		Posttest		Change		(n = 96)
	Mean	SD	Mean	SD	Mean	SD	P-value
Affect	4.31	1.00	4.79	1.01	0.48	0.95	<0.0001*
Cog. Comp.	5.14	1.01	5.39	1.07	0.25	0.87	0.0059*
Value	5.30	1.10	5.21	1.21	-0.09	1.02	0.3895
Difficulty	3.45	0.83	3.72	0.82	0.27	0.80	0.0013*
Interest	4.74	1.08	4.50	1.12	-0.25	1.08	0.0256

The first comparison to be made is that of the pre-test scores of those coming into the business statistics II course (Table 1) to the post-test historical results for the introductory course found by Schau and Emmioğlu (2012, p. 91). One would expect that the attitudes coming into a second statistics course to be similar to those coming out of a first statistics course. However, this is not necessarily the case. Value and Interest here are much higher than the post-test scores coming out of the first course, which were 4.72 and 4.00 respectively, Difficulty was much lower than what was found coming out of the first course (3.90), and Affect and Cognitive Competence were similar to the post-test historical results (4.30 and 5.03 respectively). So, students coming into the second course had higher interest levels and believed the class was more valuable than what was historically found when finishing the first course. In addition, students tended to come in thinking that this course would be more challenging. Although the difference in difficulty is not too surprising, it is curious as to why the Value and Interest components would have such large (and positive) differences.

Turning attention to the changes in attitudes during the second course (RQ1), Affect, Cognitive Competence, and Difficulty all show a positive mean change, indicating that students' attitudes became more favorable during the course. In contrast, Value and Interest were negative mean changes, indicating their attitudes became less favorable. Performing the multiple paired t-tests using an adjusted significance level of 2% ($\frac{10\%}{5}$), Affect, Cognitive Competence, and Difficulty all had significant positive changes in mean attitudes. In addition, Affect had close to a 0.5-point difference while Cognitive Competence and Difficulty were only half of that.

Because it is known that an instructor can have an effect on students' attitudes (Xu et al., 2020), it was also of interest to compare the attitudes of students in this second statistics course based on if they had the author for the first course or not. All conditions for multivariate analysis of variance (MANOVA) were assessed and were within reason. Thus, MANOVA was conducted and resulted in a p-value of 0.5316 (Wilk's Lambda) leading to the belief that there were no significant differences in mean pre-test attitudes of these components based on having had the author for the first statistics course. Similar results were found when comparing the mean attitude changes of these two groups through MANOVA (p-value = 0.8507). Thus, no further analyses were needed. The takeaway from this analysis is that having had the author for the first course had no significant effect on the attitudes in the second statistics course.

Discussion

Looking more closely at the results presented, it was first shown that over the duration of the second course, there were significant positive changes in mean attitude scores for Affect, Cognitive Competence, and Difficulty. This suggests that students, on average, significantly felt more positive feelings concerning statistics, felt significantly better about their knowledge and ability to apply statistical skills learned, and found the class to be less difficult than they thought it would be. All of these were also of greater magnitude (0.25 to 0.50 point) than what was found historically in the first course. Both Value and Interest had mean changes that were negative, but neither was significantly so. Historically, the first statistics course found a mean change of -0.32 for Value and -0.50 for Interest. In the second course here, these changes were not near as negative changes being -0.09 and -0.25 respectively. This helps show that after the second statistics course the students have less negative attitudes toward the Value of and their Interest in statistics when compared to a first course. In addition, one should note that the mean post test scores at the end of the second statistics course for four of the components are much higher than what was found historically at the end of the first statistics course. This is true for Affect (4.79 vs. 4.30), Cognitive Competence (5.39 vs. 5.03), Value (5.21 vs. 4.72), and Interest (4.50 vs. 4.00).

Out of concern of an instructor effect, follow-up analyses were conducted that compared students who had the author as their instructor of their first course to those who did not. Separate tests were then conducted on the mean pre-test scores coming into the second course as well as on the mean attitude changes over the course. Both tests showed no evidence of significant differences in these two groups. This allows us to believe that there is no difference in these two populations for how their attitudes were coming into the course or with how their attitudes changed during the course. This suggests that the positive mean changes that were found for Affect, Cognitive Competence, and Difficulty hold true regardless of what instructor the students had for their first statistics course.

Conclusion

The results here look at a second algebra-based statistics course designed to increase connections with students' careers and lives. The LDs that were included in this design were done so in a way to allow students to see how they could use statistics in their future careers while the inclusion of real data allowed for better connections to their current lives. Discussions and other assessments that were created were done so with the same goal in mind.

The investigation of students' attitudes over the duration of the course using the SATS-36 showed quite favorable findings with significant positive changes found in Affect, Cognitive Competence, and Difficulty, with no significant change in Interest or Value. All of these attitude findings are better than what has been found historically in the first statistics course, as well as in previous research. In addition, it was found that the mean attitudes at the end of the second statistics course were much higher than what has been found historically with the first course. Furthermore, it was found that the instructor for the first statistics course was not a significant factor in the students' changes in mean attitudes. Thus, it did not matter if the students had the author for the first course or not – similar attitude changes occurred regardless. Overall, the course design appeared to be a great success for student learning and for making connections to their careers and lives.

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STUDENTS' PROPORTIONAL REASONING AND MULTIPLICATIVE CONCEPTS

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Proportional reasoning is often a stumbling block for students well into college mathematics. Teachers often choose to teach their students proportional reasoning through formulas even though they themselves use a variety of strategies to solve proportional problems. This qualitative case study examines four college students' proportional reasoning on two types of proportional reasoning problems. Results showed that students who had assimilated two levels of units could apply their whole number multiplication schemes to whole number proportional problems while the assimilation of three levels of units was required to apply them to whole number and fractional proportional problems.

Literature Review

Proportional reasoning is "being able to construct and algebraically solve proportions" (Lamon, 1993, p. 41). While proportional reasoning is most often a topic of middle-grades mathematics instruction (CCSSO, 2010), it is a stumbling block for students even in college calculus (Byerly, 2019). Furthermore, many middle-grades students lack the necessary cognitive structures to construct multiplicative reasoning (Zwanch & Wilkins, 2020), let alone proportional reasoning. These findings bring to light the importance of continuing to examine students' proportional reasoning, the cognitive structures that support or limit proportional reasoning, and the ways that these limitations manifest in students' undergraduate mathematics coursework.

Fisher (1988) found that secondary teachers solve proportion problems intuitively, additively, proportionally, with a formula, or algebraically, if they answer at all. The most popular strategy chosen by the secondary teachers to teach their students when solving proportions was formulas regardless of the method these teachers used to solve proportion problems. Thus, while the same variety of strategies teachers used to solve these problems might benefit students' success with proportional reasoning, using formulas was favored in classroom instruction.

Additionally, Lamon (1993) concluded that students also use different strategies to solve proportion problems based on the semantic type of the problem. Two of the semantic problem types Lamon defined are well-chunked measures and stretchers and shrinkers. *Well-chunked measures* problems are those in which two extensive measures are being compared to form a rate. For example, in part c of the cookie problem (Figure 1), students may form a rate of $\frac{2}{3}$ cups sugar to 2 cups of flour and compare that to the rate of 1 cup sugar to $2\frac{1}{2}$ cups flour. *Stretchers and shrinkers* problems reflect "a one-to-one continuous ratio-preserving mapping … between two quantities [and] … the situation involves scaling up (stretching) or scaling down (shrinking)" (Lamon, 1993, p. 43). For example, in part a of the cookie problem, students may think about shrinking the recipe from 24 cookies to 9 and determine how the sugar will be shrunk to preserve the ratio of sugar to cookies. Specifically, well-chunked measures problems were more accessible to students than stretchers and shrinkers, and students used less sophisticated strategies when solving stretchers and shrinkers. For instance, *visual or additive reasoning*, which includes guessing and checking, answers devoid of rationale, or other incorrect intuitive solutions, were used the most frequently on stretchers and shrinkers problems. Lamon concluded that these less sophisticated strategies were likely utilized most frequently on stretchers and shrinkers problems because many students did not identify the multiplicative nature of the tasks. In combination with Fisher's (1988) finding, Lamon's conclusion that students apply a variety of strategies when solving proportional reasoning problems, particularly based on the semantic type of problem given, makes it all the more important for teachers to be prepared to understand and deliver instruction on varied strategies and ways of thinking about proportions. This qualitative case study will examine the proportional reasoning of four undergraduate students on a well-chunked measures proportion problem and a stretchers and shrinkers proportion problem.

Figure 1

The Cookie Recipe Problem

A cookie recipe calls for $\frac{2}{3}$ cups of sugar and yields 24 cookies. **[answer in brackets]**

a. How many cups of sugar would be required to make 9 cookies? **[$\frac{2}{3}$ cup]**
b. *Used only if part a was too challenging*: How many cups of sugar would be required to make 12 cookies? **[$\frac{2}{3}$ cup]**
c. Which batch of cookies is sweeter? A recipe that calls for $\frac{2}{3}$ cups of sugar and 2 cups or flour, or a recipe that calls for 1 cup of sugar and $2\frac{1}{2}$ cups of flour? **[second recipe]**

Theoretical Framework

Ulrich (2016) stated that the "assimilatory multiplicative relationship [of the third multiplicative concept] leads to the kind of immediate multiplicative reasoning needed for proportional reasoning" (p. 38). As such, the multiplicative concepts will be used as a framework to understand how students construct and coordinate units, and how students' construction and coordination of units is related to their proportional reasoning. Furthermore, Steffe et al. (2014) outlined students' construction of a proportionality scheme as a reorganization of their numerical and fractional schemes, both of which are based in part on students' units construction and coordination. Therefore, this study will examine how the cognitive structures that define students' multiplicative concepts can be used to model their solutions on well-measured chunks and stretchers and shrinkers proportion problems. In addition, this study will examine how these solutions align with Steffe et al.'s proportionality scheme.

Multiplicative Concepts

The first multiplicative concept (MC1) describes the ability to assimilate with one level of units and to construct a second level of units (i.e., *composite units*) in mental activity (Hackenberg & Tillema, 2009), therefore conceiving of a multiplicative situation in activity (Steffe, 1992). For instance, MC1 students may construct two levels of units in activity by determining that 4 groups of 7 is 28 by using their fingers, or other figurative material, to count out 7 four times. In this situation, MC1 students first conceive of seven as a single unit and then repeat that unit four times, which constitutes a second level of units.

In contrast, students with the second multiplicative concept (MC2) can assimilate with two levels of units and construct three levels of units in activity (Hackenberg & Tillema, 2009), which makes multiplicative situations assimilatory (Ulrich, 2016) or immediate. MC2 students would not require figurative material to determine that 4 groups of 7 is 28, because they can immediately conceive of the two levels of units. If asked to add an additional 6 groups of 7 to the 4 groups of 7, an MC2 student would likely determine that 6 groups of 7 is 42 and add that to 28 to find 70. The limitation of an MC2 is in their reflection on the unit structure. The third level of units decays following mental activity because it is constructed in activity. In this example, MC2 students are left to reflect only on 70 units as a structure containing 28 units and 42 units.

The third, and most sophisticated, multiplicative concept (MC3) describes a student's ability to assimilate with three levels of units and construct a fourth or fifth level of units in activity (Hackenberg & Tillema, 2009). This provides economy of reasoning. For instance, an MC3 student can immediately understand 28 as 4 groups of 7 and 42 as 6 groups of 7, can think about the total being comprised of 28 and 42, or 4 groups of 7 and 6 groups of 7, and can flexibly switch among each organization of unit structures. This is the "immediate multiplicative reasoning" (Ulrich, 2016, p. 38) that makes proportional reasoning available to MC3 students.

Proportionality Schemes

Building on the units construction and coordination that define the multiplicative concepts, Steffe et al. (2014) found that students can demonstrate proportionality at two levels. The more rudimentary is an *awareness of proportionality*. At this level, students can apply the operations of an MC2 to constitute a ratio of whole numbers as a multiplicative structure. Steffe et al. detail an example in which a student, Jill, with an awareness of proportionality is told that three cups of water and two tablespoons of lemonade powder are mixed to make lemonade. Jill is asked how much lemonade powder should be mixed with 15 cups of water. She responds, "Ten … Five times two – I don't know how that got in my head" (p. 62). Jill applied her whole number multiplicative concept (MC2) to reason about the proportions but was only ephemerally aware of how she solved the problem and solved by applying whole number knowledge. In the case of making a fractional serving (i.e., how many cups of water should be mixed with one tablespoon of powder), Jill replied, "It's like, one half cup of water, right?" (p. 61). In this fractional case, she maintains an awareness of proportionality because she halved the two tablespoons of lemonade powder. However, she could not halve the water; instead, she concluded that the recipe called for one half cup. In this statement, she conflates half of the recipe with half of a cup.

The more sophisticated level that students can construct is a proportionality scheme. Steffe et al. (2014) explain this scheme using Jack's reasoning. On the lemonade problem, Jack is asked how much powder should be mixed with one cup of water. He says, "If you had three it takes two … so that means that three halves make up one tablespoon… so you'd only need two thirds of a tablespoon to make one cup of lemonade" (p. 65). Jack's reasoning indicates a proportionality scheme because he calculated and applied a unit ratio to solve. This is evidence that he coordinated awareness of proportionality with his MC3 (Steffe et al., 2014).

This study asks, to what extent are students' multiplicative concepts related to their proportionality schemes? Additionally, it asks, to what extent can students with different proportionality schemes solve well-chunked measures and stretchers and shrinkers proportion problems?

Methods

Four undergraduate students participated in this qualitative case study. Three of the students had constructed an MC2; these students' pseudonyms are Claire, Darlene, and Phoebe. One student had constructed an MC3; her pseudonym is May. Each student participated in a semi-structured clinical interview that lasted approximately 45 minutes. The interview focused on students' proportional reasoning. The cookie task (Figure 1) will be the focus of analysis in this study. Parts a-b of the cookie task are stretchers and shrinkers problems, and part c is a well-chunked measures problem.

Results & Analysis

MC2 Students' Proportional Reasoning

Claire, Darlene, and Phoebe could not solve part a of the cookie problem. In attempting part a, they could not find a unit ratio. Claire, for example, skip counted by nines to find how many times larger 24 was than 9. When that method did not work because counting by nines does not include 24, she could not generate the unit ratio. This is one example of the counterindications of proportionality schemes generated by these students on part a of the cookie problem. These responses can also be interpreted as limitations of the students' multiplicative concepts. A unit ratio of 9:24 constitutes a three-level unit structure in which the ratio contains 24 parts, one of which is iterated 9 times. To then apply that unit ratio to the $\frac{2}{3}$ cups of sugar in the problem requires operations on three levels of units, which is unavailable to MC2 students. Thus, the lack of a proportionality scheme aligns for each of these students with limitations of an MC2.

Each of the MC2 students was successful on part b because they recognized that 12 cookies were half of 24, so they also halved the sugar. Darlene, for instance, found that the smaller cookie recipe would require $\frac{1}{3}$ cup of sugar, "Just cause it's easier for me because it's [12] half of this [24], so then you just cut that [$\frac{2}{3}$] in half [to get $\frac{1}{3}$]." However, to find half of $\frac{1}{3}$, Darlene reasoned additively, rather than multiplicatively. She said, "I brought that [$\frac{2}{3}$] up to $\frac{4}{6}$ so I could take two off of it." Darlene re-wrote $\frac{2}{3}$ as $\frac{4}{6}$ prior to determining that the smaller batch of cookies required $\frac{2}{6}$ cups of sugar, and then she equated $\frac{2}{6}$ to $\frac{1}{3}$ cups of sugar. Her language that she "could take two off of it" indicated that she conceived of $\frac{4}{6}$ as $\frac{2}{6}+\frac{2}{6}$, rather than $2 \times \frac{2}{6}$. The types of responses given by MC2 students on part b of the cookie problem provide evidence of an awareness of proportionality, as supported by the MC2 students' whole number multiplicative reasoning. Thus, while the 12:24 unit ratio is interpreted as a three-level unit structure, just as the 9:24 unit ratio was, the students leveraged their whole number multiplicative knowledge on part b to interpret 12:24 as one half. Such an interpretation is useful for MC2 students because they could disregard the two-times iteration of 12 parts contained within 24 parts that characterizes the three-level unit structure and instead conceive of the unit ratio as simply one part out of two. Thus, in situations where MC2 students could identify the unit ratio using their whole number multiplicative concepts, they were able to solve a stretchers and shrinkers proportion problem. This manner of reasoning is consistent with an awareness of proportionality and was supported by the students' MC2.

On part c, Claire and Phoebe incorrectly concluded that the first batch of cookies was sweeter. Claire could not determine which batch of cookies was sweeter because she could not find the unit ratio. She re-wrote the amounts of sugar and flour given in the problem with common denominators as $\frac{4}{6}:\frac{12}{6}$, and $\frac{6}{6}:\frac{15}{6}$, and expressed uncertainty that this was productive before explaining, "It wasn't the denominators I was trying to get right, the same, it was these numerators [12 and 15]. I'm not sure how to do that, but I still think it's the first [batch that is sweeter]." Claire's explanation indicates that she knew equating the numerators for the second fractions (i.e., the amounts of flour) would facilitate her comparison of the amounts of sugar, but her whole number multiplicative schemes did not support such a comparison. Phoebe also re-wrote the given quantities and could not identify a unit ratio, so she guessed that the first batch was sweeter. As on part a, these students were limited by their units construction because they could not construct a three-level unit structure with a unit ratio containing $2\frac{1}{2}$ cups of flour whose parts could be iterated 2 times to compare cookie recipes.

Darlene found the second batch was sweeter. Darlene also re-wrote the given quantities as $\frac{4}{6}$ and $\frac{12}{6}$, and $\frac{6}{6}$ and $\frac{15}{6}$. But, different from Claire and Phoebe, Darlene reasoned that,

> I just put the $\frac{2}{3}$ of the sugar and the 2 cups flour to have the same denominator $\left[\frac{4}{6}\text{ and }\frac{12}{6}\right]$ and then I did the same with these [1 cup and $2\frac{1}{2}$cups became $\frac{6}{6}$ and $\frac{15}{6}$]. And so, this [4 from the fraction $\frac{4}{6}$] is like $\frac{1}{3}$ of what this is [12 from the fraction $\frac{12}{6}$]. And then over here, 6 [from the

fraction $\frac{6}{6}$] can't go into 15 [from the fraction $\frac{15}{6}$] three times like this [4 and 12] can so it [the amount of sugar in the second recipe] would be over.

Darlene's approach began like Claire's, but Claire was stymied when she could not identify the unit ratio. Darlene was successful because she reasoned that the first recipe called for $\frac{1}{3}$ the sugar compared to flour, but in the second recipe, the flour to sugar was less than three times. Darlene's reasoning is consistent with an awareness of proportionality because she applied her whole number multiplicative concepts to determine that the second batch of cookies was sweeter. This was evident when she said, "6 can't go into 15 three times like this can so it would be over." Darlene reasoned that because six times three would "be over" 15, this batch of cookies had a higher proportion of sugar to flour.

MC3 Students' Proportional Reasoning

May constructed a proportional equation $\frac{2/3}{24} = \frac{x}{9}$ to solve part a of the cookie problem, and said, "One-fourth cups. … I did the scales are equal to the same factor. I just found the missing factor. … Part [$\frac{2}{3}$] to whole [24], Missing part [*x*] to whole [9]. … Like, sugar to cookies, sugar to cookies." This shows May's coordination of two three-level unit structures. The first structure contained the relationship between two-thirds and 24, and the second contained the relationship between *x* and 9. Although her method was procedural, she reflected on the proportional relationship evidenced by her retention of unit structures after activity. May's reflection on two three-level unit structures following activity is characteristic of an MC3, and supported a solution to this problem that is characteristic of a proportionality scheme.

On part c, May wrote two ratios $\frac{2}{3}: 2$ and $1: 2\frac{1}{2}$. Then, she used two concurrent proportional equations to re-scale the given recipes to 10 cups of flour. May said,

> I basically took them both and set them to where the flour would be the same amount and like proportionally whichever one had the greater amount of sugar is the one that would be sweeter. [Interviewer: And what was the amount of flour that you used for both?] 10. Because $2\frac{1}{2}$ and 2 both go into 10.

May did what the MC2 students could not; she anticipated a common referent for the amount of flour that allowed her to compare the amounts of sugar. This was supported by May's MC3 because she could maintain the three-level unit structures represented by each cookie recipe while simultaneously creating a third three-level unit structure to compare the amounts of flour, thereby identifying an appropriate common multiple, 10. As on part a, May's MC3 supported a solution on part c that is characteristic of a proportionality scheme.

Discussion

In response to the first research question, we attribute an awareness of proportionality to the MC2 students, but a proportionality scheme can only be attributed to May. Assimilating with two levels of units was sufficient to support the MC2 students' application of whole number multiplicative schemes to solve whole number proportion problems. In contrast, May leveraged her ability to reflect on three levels of units and to coordinate multiple three-level unit structures to solve whole number and fractional proportion problems. In response to the second research question, we find that the students' awareness of proportionality or proportionality scheme were applied consistently across problem types. Lamon (1993) found that students have an easier time solving some semantic problem types compared to others, and specifically, that stretchers and shrinkers problems are more difficult for students. Our results are counter to this finding, however, and indicate that the students' success was more closely tied to the numerical complexity of the problem than to the semantic problem type. Future research should incorporate all four semantic problem types and should examine students' awareness of proportionality and proportionality schemes across each semantic problem type.

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