RCML

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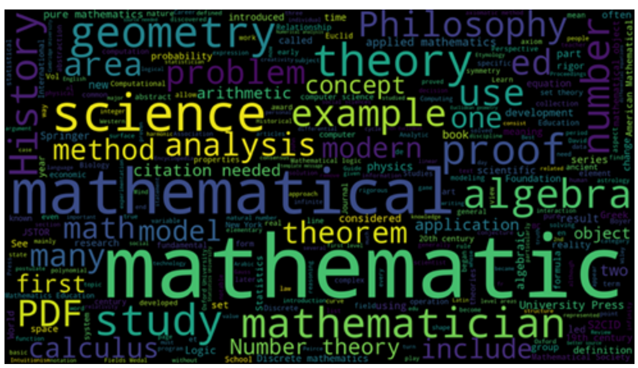

Reimagining Mathematical Learning and Teaching: Identity, Tools, and Sense-Making Across Contexts

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RCML History

The Research Council on Mathematics Learning, formerly The Research Council for Diagnostic and Prescriptive Mathematics, grew from a seed planted at a 1974 national conference held at Kent State University. A need for an informational sharing structure in diagnostic, prescriptive, and remedial mathematics was identified by James W. Heddens. A group of invited professional educators convened to explore, discuss, and exchange ideas especially in regard to pupils having difficulty in learning mathematics. It was noted that there was considerable fragmentation and repetition of effort in research on learning deficiencies at all levels of student mathematical development. The discussions centered on how individuals could pool their talents, resources, and research efforts to help develop a body of knowledge. The intent was for teams of researchers to work together in collaborative research focused on solving student difficulties encountered in learning mathematics.

Specific areas identified were:

1. Synthesize innovative approaches.
2. Create insightful diagnostic instruments.
3. Create diagnostic techniques.
4. Develop new and interesting materials.
5. Examine research reporting strategies.

As a professional organization, the **Research Council on Mathematics Learning (RCML)** may be thought of as a vehicle to be used by its membership to accomplish specific goals. There is an opportunity for everyone to actively participate in **RCML**. Indeed, such participation is mandatory if **RCML** continues to provide a forum for exploration, examination, and professional growth for mathematics educators at all levels.

The Founding Members of the Council are those individuals that presented papers at one of the first three National Remedial Mathematics Conferences held at Kent State University in 1974, 1975, and 1976.

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THE JUXTAPOSITION REGARDING LEVELS OF MATHEMATICS ANXIETY BEFORE/AFTER TEACHING EXPERIENCE

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Inservice teachers previously involved in a preservice teacher study were revisited after 5 years teaching experience to determine whether their levels of mathematics anxiety (MA) still existed and/or continued to change. A 98-item Likert-type survey, informal discussion, interviews were conducted. Findings revealed that all still experienced some degree of MA ($p < .001$). Results have implications for teacher education concerning the continued professional support of teachers, measurement of MA levels among educators, and determination of specific contexts in which MA can be interpreted and reduced.

For decades, research has documented mathematics anxiety (MA) in both pre- and inservice elementary teachers and has focused on their negative mathematical experiences, mathematical beliefs, the effect of prior teachers, and teacher education training programs (Aslan, 2013). Findings have concentrated on factors that contributed to teachers' MA and ways to address MA in preparation for teaching mathematics (Nisbet, 2015). An underlying assumption of this research is that preservice teachers with high levels of MA are likely to become teachers who do not enjoy teaching mathematics (Gresham, 2009; Iyer & Wang, 2013). Further, they may not implement effective instructional teaching practices such as those prescribed by the National Council of Teachers of Mathematics (NCTM, 2014) Principles of Action which aim to support mathematics success for all.

The quality of mathematics instruction at the K-6 level depends on the preparation of preservice teachers to teach mathematics (Nisbet, 2015) as it inevitably carries over into inservice teaching, but to what degree do mathematical confidence or beliefs, mathematical thinking, and any negative feelings/emotions experienced toward mathematics and teaching the subject continue or change? Ten inservice teachers (previously involved in a preservice teacher MA research study) were revisited to determine *whether their MA levels changed and/or to what degree has this change occurred/not occurred after 5 years teaching experience?*

Theoretical Framework and/or Related Literature

Mathematics anxiety is defined as a feeling of uncertainty and uneasiness when asked to do mathematics, has an effect on learning, and may perhaps be a greater block to mathematics learning than supposed deficiencies in our school curricula and teacher preparation programs

(Beilock & Maloney, 2015). This is cause for alarm, considering that teachers who possess higher levels of MA may unintentionally pass on these negative feelings to their students (Gresham, 2018). Studies have consistently shown that K-6 education majors have one of the highest levels of MA than those in other degreed programs (Haciomeroglu, 2013). A study by Aslan (2013), indicated that inservice teachers' had higher levels of MA than preservice teachers. He argues that differences in inservice teachers' levels of MA were related to their mathematics education and mathematical experiences. There is extensive research involving preservice teachers but relatively little is known about MA as experienced by inservice teachers over time or how to reduce it. Another study by Patkin and Greenstein (2020), suggests MA may not fade with teaching experience for those who lacked specialized training but did not track them longitudinally. Therefore, research is necessary to better understand MA and the extent to which it affects the teaching of mathematics. It is important for teachers to understand the mathematics curriculum and knowledge of effective nontraditional teaching approaches using manipulatives to bridge from concrete to abstract; implementing a variety of teaching techniques such as playing games, problem-solving strategies, small-group and individualized instruction; and addressing individuals' attitudes toward mathematics to lessen MA. This study particularly addresses a gap in literature on teacher preparation by longitudinal examination and is interested in understanding MA in relation to teaching practice.

Methodology

Participants and Setting

Ten elementary education inservice teachers, participants in a preservice study that investigated preservice teachers MA before and after a mathematics methods course were selected because even though their preservice posttest scores indicated a decrease in their MA levels those levels remained significantly high. The purpose was to determine whether their MA still existed and/or exhibited any change after five inservice years and to determine, whether any, the cause(s) of the affected change. All 10 were female. Seven were Caucasian, two were African American, and one was Hispanic. Each held a K-6 teacher certification endorsement. Three held a master's degree in education and one was within weeks of graduating with a master's degree in education. All graduate studies were completed at the same undergraduate institution. All 10 teachers were required to take undergraduate college algebra, principles in statistics, one K-6 mathematics content course, and one K-6 mathematics methods course.

Additional elective mathematics courses were available, but no one chose to enroll. To distinguish between the pre- inservice studies, each will be discussed below.

Preservice Teacher Study

This study involved preservice teachers while in a mathematics methods course. The MA Rating Scale (MARS) a 98-item, self-rating Likert-type scale developed by Richardson and Suinn (1972) was used as the pre/post quantitative instrument. The elementary mathematics methods course placed extensive emphasis upon the five process NCTM Standards: (a) communication, (b) problem solving, (c) connections, (d) representation, and (e) reasoning and proof. The course encompassed an approach to help develop an understanding of mathematics, mathematics pedagogy, hands-on approaches and strategies through modeled lessons, the use of manipulatives, and children's mathematical development, and to cultivate a positive disposition toward teaching mathematics and lessen MA. Each was required to design their own investigative lessons and were engaged in a 12-week internship experience in the schools.

Qualitative methods included structured interviews with questions from Swars et al. (2006) protocol, both formal and informal observations and discussions, and interviews initiated by the preservice teacher during or after class by the professor (the researcher in this study). The questions were adapted based on items from the MARS and the Personal Mathematics Teaching Efficacy subscale of the Mathematics Teaching Efficacy Beliefs Instrument.

Inservice Teacher Study

The MARS was used for consistency in the inservice study to identify their current levels of MA while compared with their preservice teacher study posttest score to determine whether any change occurred and/or continued to be affected after 5 years teaching experience. Interview questions and prior preservice comments were used to identify perceptions of their own skills and abilities to teaching mathematics effectively as well as to how their MA may have affected these perceptions before and after teaching experience.

Data Analysis

Descriptive and inferential statistics and paired sample t tests were completed to consider differences between pre- and posttest anxiety levels from both studies. The posttest MARS score from the preservice study was subtracted from the inservice MARS score to reveal a difference in scores. The qualitative data were analyzed both individually and collectively using a grounded theory approach. Selective coding processes were used to integrate categories to establish a generalized framework, with subsequent themes.

Findings

Quantitative Findings

Table 1 illustrates preservice teachers' MA scores from the previous study. Scores revealed significant decreases in all 10 teachers' MA from pre- to posttest after participating in a mathematics methods course. Even though their MA levels did decrease, their overall total MARS score remained significantly high, thus ranking them in the "high MA group" quartile.

Table 1

MARS Preservice Pretest/Posttest Comparison Scores

Teacher	Grade level	Preservice Pretest Score	Preservice Posttest Score	Gain
Teacher 1	First	331	260	−71
Teacher 2	Second	343	269	−74
Teacher 3	Second	273	212	−61
Teacher 4	Second	297	218	−79
Teacher 5	Third	328	250	−78
Teacher 6	Fourth	299	231	−82
Teacher 7	Fourth	296	228	−68
Teacher 8	Fourth	300	216	−84
Teacher 9	Fifth	275	202	−73
Teacher 10	Fifth	283	205	−78

Note. MARS = MA Rating Scale; $p < .005$.

Table 2 shows the comparison of the preservice posttest–inservice test raw mean scores from the inservice teacher study. It was found that overall inservice teachers' MA (although still highly prevalent) decreased slightly after 5 years teaching experience.

Table 2

MARS Inservice Comparison Scores

Teacher	Grade level	Preservice Posttest Score	Inservice Test Score	Gain
Teacher 1	First	260	251	−09
Teacher 2	Second	269	261	−08
Teacher 3	Second	244	229	−15
Teacher 4	Second	238	223	−15
Teacher 5	Third	250	209	−41
Teacher 6	Fourth	231	204	−27

Teacher 7	Fourth	228	202	−26
Teacher 8	Fourth	216	199	−27
Teacher 9	Fifth	202	186	−16
Teacher 10	Fifth	205	192	−13

Note. MARS = MA Rating Scale, $p < .001$.

Qualitative Findings

Significance of the Mathematics Methods Course(s)

All 10 teachers commented that their MA was consistently evident during the last 5 years. Even though Teacher 5 had significant decreases in scores in both studies, she expressed her lack of mathematical confidence in part due to her MA. She also implied that earning a master's degree in education significantly enhanced her mathematical knowledge base and was a factor towards change in her MA. Teachers 6, 7, and 8 (each earning a master's degree in education by their fourth inservice year) also had significant decreases in scores after teaching experience. All three reiterated that mathematics methods courses (taken with the same professor as their undergraduate) and the required action research which offered time to reflect and change instructional teaching practices including addressing MA and professional development mathematics workshops in their graduate program of study positively affected their attitudes, MA, and mathematical confidence; thus, improving their overall mathematics knowledge. However, Teachers 5, 6, 7, and 8 (all four master's teachers) expressed the need for "career long" mathematics professional development to continue strengthening their mathematics skills and effective mathematics teaching practices while specifically addressing and/or alleviating their MA. Teachers 3, 4, 9, and 10 (who did not have a graduate degree) expressed a strong need for further mathematics professional development and mentorship within their schools to help with their educational needs. However, each expressed they did not want to take additional mathematics courses to address their mathematical deficiencies.

Attitudes Towards Mathematics

All 10 teachers felt their MA decreased as they gained more teaching experience. However, each indicated they did not like mathematics and really struggled with the subject as negative attitudes were still evident five years later. They described being "stressed," "embarrassed," "frustrated," "fearful," "discouraged," and "struggling" but expressed being more comfortable using nontraditional teaching practices and manipulatives with teaching experience. Even though Teachers 1 and 2 were certified to teach K-6, both revealed that they felt more comfortable teaching the lower grades because they lacked mathematical content confidence to teach in the upper grades and could not effectively implement mathematics lessons. Each indicated they would quit the profession if moved to upper grades. Teacher 1 taught grade levels fifth, fourth, and third respectively, and finally first grade within her 5 years of teaching. Teacher 2 noted that she was hired to teach fifth grade initially, but after 3 weeks in the classroom realized her MA was too overpowering and was moved to second grade. It is noted that teachers with the highest levels of MA taught in the lower grades and wish to remain there.

Demands of Teaching

Interview comments highlighted that all recognized the need for mathematical knowledge in making meaningful contexts, which are constructed, connected, and applied to mathematical learning, and with actively engaging students. Each described being overwhelmed by the challenges of teaching. They felt unprepared and unsupported with the demands of teaching, had overwhelming concerns with ongoing commitment of time and energy, needless paperwork, lesson planning, preparing materials, the environment, management skills, lack of leadership and isolation in the ability to express publicly their levels of MA for fear of losing their jobs.

Discussion and Conclusion

Even though the decrease was minimal, results concluded that teachers' MA was reduced through their teaching practice. We are reminded through the teachers' voice of how teacher mastery of the mathematical content also has an effect on the students (Beilock & Maloney, 2015) as it goes back to teacher preparation and knowledge of subject matter, and the need for professional development and administrative supports. The usefulness and experience of the mathematics methods course was a salient finding. Teachers longed to situate the course within development as an elementary educator. We want mathematics methods courses adequately preparing teachers for the long haul and for the demands of teaching, which also includes addressing, alleviating, and/or reducing MA not only within themselves but also in their students. Even though their MA did slightly decrease after teaching experience, the greatest decrease in MA occurred while enrolled in the preservice mathematics methods course. This research provides durability and supports the importance of methods courses, collaborative experiences, and the use of manipulatives in shaping teaching practices and reinforces compelling evidence as illustrated by Aslan (2013) and Haciomeroglu (2013) that teachers are less successful in conveying important mathematics concepts that are requisites for students' academic growth as the cycle of MA continues with teaching experience. Four of 10 teachers chose to address their ineffective practices, mathematical confidence, and content knowledge by completing a master's degree in education within the first five years of teaching, hoping it would continue to lessen their MA. Those who completed their master's degrees also had the largest decrease in MA levels. A plausible reason for the decrease may be due to the continued development of mathematical knowledge. Some teachers seemed to be more willing to explore various avenues to improve their knowledge of mathematics by obtaining a master's degree and concluded their involvement in graduate studies lessened their MA. Teachers are an important influence not only on the quality of an individual's mathematical learning but also with their own learning as well. However, not all those involved completed graduate studies, so no conclusion can be drawn as to the effectiveness of more striking experiences in graduate work.

All proffered continued professional development (workshops that specifically addressed MA and mathematics content) were a needed career-long designation. Inservice professional development could broaden teachers' conceptual understanding of mathematics, which, in turn could not only lessen or reduce their MA but also students' anxiety as well. More research is needed to investigate inservice continuing education and professional development and its effectiveness in lessening or reducing MA.

This study has implications for teacher education programs, particularly mathematics methods courses. Teachers who are reluctant to broaden their mathematical knowledge, perhaps due to MA may not be interested in learning about alternative teaching methods that could help students learn mathematics. These teachers tend to use traditional methods and the same lesson plans that they have developed over time. Therefore, K-6 teachers must learn how to incorporate a variety of effective teaching methods and practices to meet each student's unique needs. Teachers who understand the learning needs of others are more empowered to provide the kind of instruction their own students need as was evidenced in this study. There is no doubt that for some mathematics is and will remain challenging. The study's qualitative findings show many lacked mathematics confidence; however, teachers should discard false beliefs and intimidation due to lack of confidence, for the constructs of MA has a profound effect on learning and the potential to become effective teachers. Even though this study is situated within a smaller sample size, the argument has demonstrated the complexity of MA as a universal concern for all mathematics educators. When considering the findings, a determination is not made that changes in MA will continue. However, carefully examining the process of change even with this study's sample size may help us become better informed not only about the longitudinal effectiveness of our mathematics methods course but also about the usefulness in understanding the important outcomes of those mathematics methods courses across time and, thus, eliminate the cycle of mathematics anxiety.

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AI-MEDIATED MATHEMATICS PERSONIFICATION: PRESERVICE TEACHERS' IDENTITIES REIMAGINED

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This study explores how preservice teachers (PSTs) use personification and AI-mediated reflection to examine their evolving emotional and professional relationships with mathematics. Building on our prior studies (Howell et al., 2025; Tîrnovan et al., 2025), participants personified mathematics, created dialogues, and reflected on AI-generated interpretations. Findings reveal reconciliation, empowerment, and resilience as PSTs reimagined mathematics as a relational mentor. AI served as a reflective mirror, deepening self-awareness and emotional insight. Results highlight AI-mediated personification as a promising approach for fostering identity formation and reflective practice in mathematics teacher education.

Introduction

Teachers' relationships with mathematics are deeply emotional and culturally situated. Through challenge, discovery, and reconciliation, these relationships shape teachers' identities regarding learning, teaching, and professional growth (Hannula, 2012; Sfard & Prusak, 2005). While identity formation emerges through affective and social negotiation (Lutovac & Kaasila, 2014), critical consideration of how preservice teachers (PSTs) emotionally experience mathematics remains underexamined.

Personification activities provide an imaginative and affective means for PSTs to anthropomorphize mathematics (e.g., friend, mentor, rival, or companion), creating an inviting space for emotional disclosure and self-awareness. They can surface deep-seated attitudes toward mathematics, reveal sources of anxiety or empowerment, and support reimagined pedagogical identities (Hodgen & Askew, 2007; Zazkis, 2015).

This study introduces artificial intelligence (AI) as an interpretive and reflective partner in PSTs' mathematics personification processes. When designed with anthropomorphic and empathetic features, AI systems can foster user engagement, reflection, and emotional resonance (Ackermann et al., 2025; Janson, 2023). By incorporating AI-generated images and textual interpretations into the personification assignment, AI can act as a semiotic mirror and amplifier of PSTs' emotional mathematics narratives (Heinsfeld & Veletsianos, 2025; Joseph, 2025).

This study aims to contribute to conversations at the nexus of narrative imagination, emotion, and AI in mathematics teacher education. The following literature review situates this study within scholarship on mathematical identity, personification, and AI as a reflective mediator.

Literature Review

Past mathematical experiences shape teachers' pedagogical identities. While Hodgen and Askew (2007) confirm this among primary teachers, Lutovac and Kaasila (2014) contextualize preservice teachers' future-oriented identity work through narrative reflection on past mathematical experiences. Personification has proven valuable for examining teachers' mathematical identities and uncovering patterned emotional trajectories. Howell et al. (2025) report that preservice teachers who personified mathematics as supportive friends or patient mentors reconciled past struggles into renewed confidence. Extending this work to in-service teachers, Tîrnovan et al. (2025) identify a cycle of estrangement, reconciliation, and empowerment that paralleled pedagogical development. In both studies, teachers' reconciliation with mathematics fostered empathy toward students experiencing similar struggles.

Emerging AI research suggests the potential to deepen such reflective processes. Studies demonstrate that AI systems with anthropomorphic features (human-like appearance or communication) enhance user engagement and emotional connection (Ackermann et al., 2025; Janson, 2023). In teacher education, Karataş and Yüce (2024) report that PSTs who engaged with AI tools experienced heightened self-awareness and reflection on their practice. Heinsfeld and Veletsianos (2025) observe that AI is increasingly understood as a mirror for human thought, while Polyportis and Pahos (2024) find that trust in educational AI is linked to its perceived human-like qualities.

While these studies illuminate how anthropomorphic AI fosters engagement and reflective awareness, none examine AI as an interpreter of learners' identity narratives. How learners respond when AI receives their self-stories and returns visual and textual interpretations has received little attention. This study addresses that gap by integrating AI-generated imagery into a personification assignment, examining how PSTs respond when their self-narratives are returned in transformed form.

Theoretical Framework

This study is grounded in narrative identity theory and affect-identity integration. Sfard and Prusak (2005) conceptualize identity not as an internal essence but as discursive activity: a collection of reifying, endorsable, and significant stories about people. They distinguish actual identity (stories about current states) from designated identity (stories about hoped-for states), with the tension between these driving developments. Kaasila (2007) demonstrates that mathematical biographing is reconstructive, as teachers integrate past experiences into emerging professional identities. Thus, narrating is constitutive of identity discovery and development rather than merely descriptive.

Personification invites PSTs to produce identity narratives by storying their relationships with mathematics (Zazkis, 2015). When PSTs personify mathematics as a friend, mentor, or adversary, they surface beliefs and emotions that typically remain unarticulated. Hannula (2012) argues that emotion is constitutive of mathematical identity rather than separate from cognition. The affective content of personification narratives, including fear, reconciliation, and empowerment, is identity-in-formation rather than a description of feeling.

AI-generated textual interpretations of PSTs' narratives and translations into new textual and visual forms provide a novel, significant narrative interpreter and narrator, externalizing PSTs' relationships with mathematics (Joseph, 2025). PSTs' evaluations of AI's externalized perspectives result in validation, resistance, or revision, all of which constitute identity work. We examine: (i) how PSTs personify their relationships with mathematics, and (ii) how AI-generated interpretations prompt PSTs' validation, resistance, or revision of those narratives.

Methodology

Participants included 14 elementary PST enrolled in a mathematics content and pedagogy course at a public university in the southeastern United States. All participants had previously completed general education mathematics courses and were preparing for student teaching. The group represented diverse cultural and linguistic backgrounds, including South Asian, Middle Eastern, Latin American, African American, and Anglo-American heritages. To maintain both confidentiality and fidelity, we assigned culturally consistent pseudonyms to each participant.

We drew data from our three-part *Mathematics Personification Assignment*. Part I, the Personification Narrative, asked participants to personify mathematics and describe their relationship with mathematics. Part II, the Dialogue with Math, asked participants to write a creative conversation between themselves and their mathematics-personified form, infusing it with voice and tone. Part III, the AI-Mediated Reflection, introduced AI-generated interpretations of PSTs' personification narratives and dialogues. Using ChatGPT or Gemini, PSTs submitted their narratives and dialogue using standardized prompts: one requesting a visual representation that captures the emotional tone and symbolism of their math character, and another requesting an analysis of the relationship's emotional patterns. PSTs then reflected on how these interpretations aligned with their self-perception.

We merged the four-fold results (personification narrative, dialogue with math, AI-generated interpretations, and PSTs' reflections) to create a comprehensive transcript for each participant. Maintaining methodological continuity, analysis followed the same hybrid deductive-inductive coding process in our prior personification studies (Howell et al., 2025; Tîrnovan et al., 2025). Herein, this was expanded to include AI-Enhanced Interpretations, capturing how AI-generated interpretive visualizations influenced participants' emotional and professional reflections.

Coding proceeded in four stages. First, we holistically reviewed each transcript to identify emergent relational and emotional patterns. Second, we coded the data using established categories, including *Evolution of Relationship*, *Emotional Evolution*, *Power Dynamics*, and *Professional Practice*. Parts I and II used the same prompts and coding categories as Howell et al. (2025) and Tîrnovan et al. (2025). Since Part III introduced the AI-mediated innovation, we coded participants' reflections specifically to AI engagement (e.g., *AI-Reinforcement*, *AI-Empathy*, *AI-Transformation*). Last, we summarized code frequencies and intersections in a cross-participant matrix to illuminate convergent themes such as mathematical resilience, empowerment, and professional identity development.

Building on our previous work, we approached AI as both analytic support and interpretive collaboration with confidence in its reliability and transparency. While all interpretive judgments (e.g., iterative review and cross-checking for internal consistency) ultimately remained researcher-driven, AI facilitated the consolidation of the four-part dataset and ensured consistent application of the coding framework. Transparent analytic documentation and repeated code validation led to analytic trustworthiness. We approached the analysis reflexively, recognizing the interpretive influences of our identities as mathematics educators on participant narratives. All procedures met institutional ethical standards for research with human participants.

Findings

Parts I and II: Personification Narratives and Dialogues with Math

Analysis of the fourteen PST personification narratives revealed that every participant expressed an evolving, affective relationship with mathematics. PSTs personified mathematics as an active presence reflecting emotional history, struggle, and growth, narrating their relationships as *in progress* rather than resolved, and their identity as still in formation. Their narratives represented *active reparation*, revealing their years-old K-12 emotional mathematical wounds as still healing.

PSTs' personification narratives prominently represented a pattern of reconciliation with mathematics, accompanied by feelings of empowerment and restored confidence. Participants, like Sadia Khan and Linh Nguyen, described overcoming past fear and rebuilding trust and wrote of mathematics as a returning friend. Thus, reconciliation with mathematics is an emotional and professional developmental process.

Across participants, mathematics emerged as a relational mentor (wise, patient, and emotionally attuned). While Lucía Torres imagined mathematics as "that eccentric professor," Ellen Carter depicted her as "an older relative" who teaches through patience rather than authority. These depictions align with our power-related codes, Mathematics as Friend and Mathematics as Authority, suggesting that PSTs increasingly view mathematics as a positive rather than an adversarial relationship. The PSTs' viewing mathematics as a *guide or senior mentor* rather than as a professional colleague underscores that PSTs still imagine themselves in an apprenticeship, emotionally dependent on a relational mentor.

Some PSTs framed mathematics through creativity, cultural connection, and community imagination. Tiana Brooks revived *Mathia*, her childhood friend who turns "a pile of blocks into a story," blending artistic play with cognitive reasoning. Similarly, Camila Rosario envisioned mathematics as a community collaborator, designing a "summer camp to help kids love you instead of fear you." These narratives activated the sociocultural codes Mathematics as Cultural Mediator and Mathematics as Community Builder. PSTs narrated aspirational practices, imagining community and creativity as goals of their future classrooms. Thus, personification became both a reflection and a projection, articulating who they hope to become as educators.

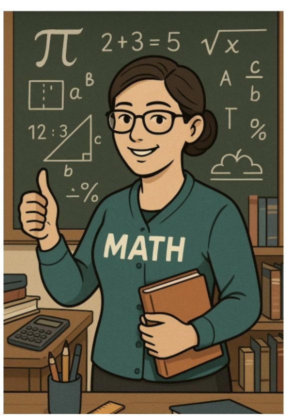
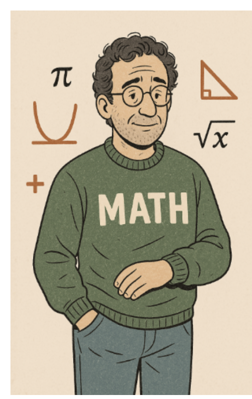
PSTs consistently framed their relational struggle with mathematics as essential to growth. They expressed resilience and emotional regulation through metaphors of patience, process, and persistence. Maha El-Sharif wrote, "He taught me to be patient and to trust the process." PSTs described resilience as a personal coping mechanism to rebuild confidence in learning mathematics. By valuing struggle and relational resilience, PSTs demonstrate emotional awareness and the capacity to transform challenges into effective pedagogy.

Part III: AI-Mediated Reflection

The integration of AI-generated interpretations of PSTs' mathematics personification narratives in Part III elicited a new dimension of insight. Figure 1 provides examples of these artistic interpretations. (Examples from Participants 13 and 14 are not included because they did not submit AI-generated images for analysis; their written reflections aligned with images shown in the Appendix.) The codes AI-Reinforcement, AI-Empathy, and AI-Transformation appeared in every case, suggesting that AI served as both a mirror and a catalyst for pedagogical change.

Figure 1

AI-Generated Interpretations of Psts' Mathematics Personification Narratives

 Participant 1	 Participant 2	 Participant 3
 Participant 4 (similar to Participants 6 and 12)	 Participant 5 (similar to Participants 7 and 9)	 Participant 8 (similar to Participants 10 and 11)

The PSTs' engagement with AI as an interpretive partner was accepting and affective, deepening awareness of their evolving teacher identities. In response to the AI-generated imagery, they frequently expressed relational feelings of empathy and recognition, with several noting that the AI *saw* and *understood* them. They reported feeling validated, surprised, and even transformed by how the AI visualized their personified math. For instance, Miguel Herrera stated, “The AI’s image made me smile. It captured exactly how I felt.” Lucía Torres reflected that “Seeing Math rendered this way helped me realize how much empathy I’ve developed.” As the PSTs attributed human-like understanding to AI, they felt their emerging identities validated. Thus, AI served as both an emotional mirror and a pedagogical bridge as AI reflections helped PSTs visualize affective and professional dimensions of their relationship with mathematics.

Synthesis, Implications, and Conclusion

Our findings suggest that AI-mediated personification reflections can be powerful tools for developing PSTs’ relational and emotional awareness of mathematics. Through creative narrative and AI-interpretive visualization, PSTs reframed struggle as growth and saw mathematics as an empathetic collaborator rather than an impersonal system. AI-mediated reflection helped PSTs to construct evolving narratives of reconciliation, empathy, and resilience. While mathematics was personified as a relational force, a mentor, or a mirror through which self-understanding could emerge, the integration of AI-generated imagery helped PSTs visualize emotional growth, translating such into pedagogical intention. Embedding such activities into teacher preparation coursework may help future educators identify emotional barriers, cultivate resilience, and align their personal mathematical histories with emerging pedagogical philosophies. This work contributes to a broader understanding of how emotional and technological tools can support identity formation across the continuum of mathematics teacher education. These findings should be interpreted within the scope of the study's design. The small sample of PSTs drawn from a single course, along with variability in participants’ use of AI platforms and outputs, suggests the need for future research across larger, multi-site contexts and more systematic comparisons of AI tools and modes of engagement.

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REIGNITING JOY, AGENCY, AND BELONGING: HOW MATH TEACHERS' CIRCLE SUMMER IMMERSION EMPOWER EDUCATORS

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This study explores survey data from 2021 to 2025 to examine how the Math Teachers' Circle Summer Immersion (MTCSI) program shaped teachers' mathematical growth, instructional practice, and professional identity in post-pandemic contexts. Analysis of responses from 363 participants revealed three overarching themes: Affective and Identity Growth, Collaborative Belonging and Social Learning, and Agency and Pedagogical Transformation. Teachers reported increased confidence, creativity, and commitment to inquiry-based, student-centered instruction, alongside strengthened professional identity and community, while noting challenges such as limited time, resources, and administrative constraints in implementation.

Introduction

Professional development (PD) plays a vital role in enhancing teachers' mathematical knowledge, refining their pedagogical skills, and fostering instructional practices that promote deep conceptual understanding (e.g., Ball et al., 2008, White et al., 2013). The MTCSI workshops exemplify an effective PD model that strengthens both content knowledge and instructional practice, through rich, nonroutine problem solving that encourages creativity, persistence, and reflection while modeling effective inquiry-based teaching (White et al., 2013). Guided by experienced facilitators, teachers work through challenging mathematical tasks and reflect on questioning, facilitation, and perseverance as key instructional practices. These experiences enable teachers to connect their own problem-solving processes to classroom strategies that promote student-centered learning, building confidence in making problem solving a central component of instruction. This study examines how MTCSI participation shapes teachers' professional identities and instructional practices by reigniting joy, curiosity, and agency in mathematics teaching amid competing instructional demands.

Literature Review

Immersive Professional Development

Immersive PD like MTCSI engages teachers as active participants in mathematical inquiry with mathematicians, fostering creativity, collaboration, and deeper understanding. Research shows this approach strengthens teachers' mathematical knowledge and inquiry-based instruction while building sustained professional communities that bridge K–16 contexts (White et al., 2013). As a community of practice, MTCSI reduces isolation and supports long-term growth in teachers' professional identities (e.g., Renga et al., 2020).

Rural and Underserved Contexts

Access to high-quality PD remains uneven across geographic and socioeconomic contexts, leaving many rural and underserved mathematics teachers with limited opportunities to enhance their practice (Ball et al., 2008; Howley & Howley, 2005). Funding barriers, travel constraints, and limited locally relevant PD often led to professional isolation, particularly in small schools lacking peer collaboration. This isolation hinders reflection and contributes to teacher attrition (Ingersoll & Tran, 2023). Community-based PD models like MTCSI help address these inequities by fostering collaboration, collegial support, and meaningful professional networks.

Content Knowledge, Pedagogical Growth, and Affective Engagement

Research on mathematics teacher PD emphasizes strengthening both content knowledge and pedagogical practice (e.g., Ball et al., 2008), yet emerging work highlights that effective teaching also depends on teachers' emotional engagement with mathematics itself. Scholars have shown that cultivating curiosity, joy, and intrigue in mathematics lessons enhances engagement and deepens understanding for students (Dietiker et al., 2023), and similar affective experiences for teachers can inspire more student-centered, inquiry-based approaches that promote persistence and productive struggle (e.g., Bolyard et al., 2023; Stipek et al., 2001).

Professional Identity and Teaching Practice

Teachers' professional identity, how they interpret and enact their classroom roles, is shaped by their beliefs, experiences, and community (Renga et al., 2020). PD fosters emotional connection to mathematics and positions teachers as problem solvers (Hodgen & Askew, 2007). Further studies (Hobbs, 2012) suggest that aesthetic experiences in mathematics enhance confidence and agency. Yet, few PD models integrate affective and identity renewal through problem solving, underscoring value of MTCSI's blend of exploration, emotion, and community.

Rationale for the Present Study

Studies on Math Teachers' Circles (e.g., White et al., 2013) demonstrate that sustained, community-based PD strengthens teachers' confidence, and mathematical content and pedagogical knowledge. Despite these benefits, access to such PD remains inequitable in rural and underserved contexts. For over a decade, our MTCSI has addressed this gap by providing immersive PD for teachers in rural and underserved urban schools, emphasizing renewed disciplinary curiosity and mathematical identity. This study examines how MTCSIs foster teachers' enthusiasm for mathematics, collaborative mathematical problem solving, and shaping their conceptions of effective instruction in post-pandemic contexts (2021–2025). It also explores the challenges teachers face in these settings when enacting such practices in their classrooms.

Theoretical Framework

This study is grounded in social constructivism (Vygotsky, 1978), which views learning as a shaped social process through dialogue, collaboration, and shared meaning-making. Teachers' understanding develops not in isolation but through interaction with others who challenge and extend their thinking. The MTCSI program exemplifies this foundation by engaging teachers and mathematicians in nonroutine problem solving within a collegial environment where alternative strategies are valued and mistakes are treated as learning opportunities. From a socio-constructivist perspective on affect, learning is inseparable from emotion and identity, as emotions like joy are constituted through participation in problem-solving practices and shaped by individuals' appraisals of themselves, the task, and the social context (Op't Eynde, 2006).

Building on this foundation, the study draws on two interrelated perspectives: communities of practice (CoP; Wenger, 1998) and socio-constructivist theory of affect and identity (Op't Eynde, 2006). Within CoP, learning occurs through shared participation as members negotiate meaning and build collective expertise. Emotions like joy are understood as situated responses that emerge through participants' engagement, appraisal, and sense of belonging within the community. The MTCSI program embodies this through collaborative mathematical inquiry that supports deeper understanding, foster belonging, and professional identity within a community.

Methodology

This study adopts a social constructivist perspective, positioning teachers as active learners who build understanding through experience, reflection, and collaboration. The MTCSI, a three-day summer workshop in southeastern U.S. state, is designed around sessions focused on inquiry-based learning and productive struggle. It engages teachers in rich, nonroutine problem solving to rekindle curiosity, creativity, and joy while fostering collegial reflection. Its community-based format promotes collaboration among K–16 educators across grade levels and contexts, strengthening professional connections and engagement in mathematical inquiry.

Participants

This study included 363 K–16 teachers who participated in the MTCSI program between 2021 and 2025. Nearly 70% of participants taught in rural or underserved urban schools. Teaching experience ranged from early careers to veteran educators, with 45% having 0-5 years, 26% between 6-19 years, and 28% more than 20 years of experience.

Structure of a Typical Session. Each session began with an open-ended, low-floor, high-ceiling task, such as fairly dividing two loaves of bread among three people, which led to exploring Egyptian fractions, expressing a share as a unit fractions (e.g., each person receives $\frac{1}{2}+\frac{1}{6}$ of a loaf). Accessible yet challenging, these activities engaged teachers intellectually and emotionally, helping them empathize with students' experiences and envision how inquiry-based mathematics could transform practice. At the end of each MTCSI, participants completed anonymized surveys including 3-point Likert-scale (0 = Disagree, 1 = Agree, 2 = Strongly Agree) and open-ended questions (see Results) about their mathematical experiences, professional growth, and instructional impact.

Data Analysis

Open-ended responses were analyzed using a reflexive inductive and deductive thematic approach (Braun & Clarke, 2022). During initial open coding, we attended to participants' descriptions of experiences, actions, and evaluations of their MTCSI participation. Rather than coding only for explicit emotion words, codes captured evidence of affect through teachers' descriptions of energy, renewal, enjoyment, frustration, perseverance, belonging, and self-positioning as learners (e.g., *renewed energy*, *felt challenged*, *part of math community*). Two researchers conducted this phase independently. These inductive codes were then examined deductively through communities of practice and a socio-constructivist theory of affect. Consequently, related codes were iteratively grouped into subthemes and organized into three overarching themes through a consensus-building process that refined code definitions.

Findings and Discussion

Quantitative Results on Instructional Practice and Classroom Impact

Survey data demonstrate consistently positive perceptions of the impact of MTCSIs on instructional practice. Across eight Likert-scale items, majority of participants reported strong agreement (percentages in parentheses) that MTCSI increased personal interest in mathematics (81%), enhanced classroom creativity (83%), improved student engagement (76%) and understanding (66%), encouraged student-centered instruction and greater time for problem solving (both 69%), increased access to instructional resources (73%), and emphasized productive struggle (84%). No participants selected "disagree" for these items, reflecting a clear consensus about the benefits of MTCSIs. Overall, findings highlight MTCSI as an effective PD that promotes instructional innovation, deeper mathematical engagement, and renewed identity.

Qualitative Themes

We analyzed responses to open-ended questions about MTCSI's impact on teachers' mathematical growth, instructional practice, and professional perspectives. Two additional questions explored future applications and challenges. The analysis revealed three overarching themes with eight interrelated subthemes illustrating how MTCSI fostered joy, community, and renewed teacher agency.

Affective and Identity Growth

This overarching theme captures how MTCSI participation reignited teachers' enthusiasm for mathematics and reaffirmed their professional identities as confident, joyful educators.

Reigniting Joy and Purpose in Mathematics Teaching. Participation in an MTCSI rekindled teachers' enthusiasm for mathematics, renewed their professional identity and passion for teaching. Joy was evidenced through renewed energy, cognitive stimulation, and motivation, not emotion words alone. Many described the experience as refreshing and transformative: "Renewed energy and teaching vigor again after the pandemic," and "Parts of my brain that have lain dormant since college feel woke up-the stimulation was refreshing and so needed." Another teacher reflected, "My soul is happy because I got to nerd out with all my math peeps!"

Embracing Productive Struggle and Mathematical Growth. Teachers valued the intellectual challenge of open-ended problem solving, noting that "the sessions exposed me to open-ended ways of approaching math and the value of productive struggle." Others shared that "it was good for me to struggle a little" and that "being challenged helps me grow mathematically and professionally." These experiences reframed struggle as an essential part of learning, fostering perseverance, curiosity, and deeper engagement with mathematics.

Renewal and Professional Well-Being. For many teachers, the MTCSI served as both rejuvenation and self-care, offering emotional renewal after demanding school years. One participant shared, "This is my annual reset. I missed last year & it was my toughest year ever! Camp gives me a mental restart," while another added, "Rejuvenating my love for math is excellent self-care." These reflections show MTCSI as a space of healing and renewal, strengthening teachers' resilience against burnout.

Teachers' Learning Through Teacher-as-Learner Positioning in a Professional Community

This theme captures how teachers, positioned as learners within a community, developed shared instructional practices, reflexive empathy, and understanding of vertical alignment in math standards consistent with CoP and a socio-constructivist theory of affect and identity**.**

Collaboration and Community Building. Nearly all respondents highlighted connection, teamwork, or shared inquiry. One teacher wrote, "The collaboration, the community-building, hospitality and setting were out of this world!" Others celebrated "being a part of the math community and meeting others to collaborate and glean information." These comments indicate active participation in a shared professional space rather than isolated learning.

Reflective Empathy and Student Perspective. Collaborative learning fostered perspective-taking as teachers' own struggles mirrored those of their students: "Doing computations in public, making errors, it was embarrassing. I see how students feel." Such shared vulnerability nurtured empathy and mutual respect.

Linking Mathematical Ideas Across Grade Levels. Working with colleagues across grade levels expanded teachers' understanding of vertical alignment in mathematics. **"**I loved seeing the connections between all levels of math as we worked, elementary to university level." Many teachers noted, gaining insight into how ideas progress across grades and strengthening a sense of belonging within the MTCSI community.

Professional Identity, Agency and Pedagogical Transformation

This theme captures how teacher agency developed through social participation, affective appraisal, and shifts in professional identity. Consistent with a CoP framework, instructional change emerged as teachers constructed new meanings by doing mathematics, reflecting on facilitation moves, and continually negotiating tensions between disciplinary practices and pedagogical demands as they developed useful knowledge for teaching. This process supported agentic decisions to experiment, take leadership, and enact student-centered instruction. From a socio-constructivist affective perspective, positive emotions strengthened professional identity and agency, enabling meaningful pedagogical transformation. This theme extends Renga et al.'s (2020) work by showing how renewed teacher identity, developed through negotiation within a professional community, supported instructional transformation and teacher agency.

Pedagogical Reflection and Transfer to Practice. Teachers frequently connected workshop experiences to classroom implementation: "I loved the idea of using vertical surfaces and only one pen so that thinking is shared." Others shared, "It's given me new ideas on how to take something seemingly simple and use it to teach problem solving and deductive reasoning skills," As one noted, "Every year I'm encouraged to provide more work time for students … These tasks allow them to explore math in a safe space." These reflections indicate a shift toward facilitative, student-centered teaching that values reasoning, collaboration, and exploration.

Mathematical Confidence and Agency. Many expressed renewed confidence, "Maybe I know more math than I give myself credit for", and ownership over their professional growth. Some reported leading math circles or PD sessions, while others planned to adapt open-ended challenging problems in their districts.

Reimagining Teaching Practice Through Mathematical Exploration. Engaging in doing mathematics redefined teachers' views of instruction. One teacher shared, "It reminded me to think creatively, the same habits I want my students to develop". Another reflected on "how to present problems so students discourse is encouraged, even for shy or anxious students." These insights show how inquiry-based learning fosters creative and inclusive teaching.

Challenges in Transferring MTCSI Learning to Classroom Practice

Teachers valued the MTCSI experience but noted challenges in applying it. About 33% cited time and testing pressures: "giving students the time to problem solve but feeling the pressure of the reality of time crunch." Others mentioned limited resources or administrative barriers, such as "My admin is so focused on my use of their curriculum that I cannot implement these ideas beyond sneaking them in." One teacher summarized, "Don't force curriculum coverage, teaching thinking skills first will let you cover more with more students." Overall, these reflections show that while MTCSI reignited enthusiasm for exploratory math teaching, teachers still face systemic limits of time, resources, and policy.

Conclusion

This study contributes to understanding how immersive PD can foster teachers' mathematical joy, agency, and instructional renewal. The MTCSI program fostered meaningful professional growth by combining mathematical exploration, collaboration, and reflective practice. Teachers reported increased confidence, creativity, and commitment to inquiry-based, student-centered instruction, alongside strengthened professional identity and agency. While they found deep value in creativity and inquiry-based learning, many struggled to implement these approaches fully amid the pressures of pacing, testing, and administrative expectations. The MTCSI program provided space for teachers to discuss such challenges and identify flexibility within policy constraints to sustain inquiry-based mathematics learning.

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CULTURALLY RESPONSIVE TEACHING: PERSPECTIVES OF INTERNATIONAL MATHEMATICS GRADUATE TEACHING ASSISTANTS

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This study examines how international graduate teaching assistants (ITAs) in mathematics, with their valuable cultural perspectives, implement culturally responsive teaching (CRT) in U.S. undergraduate mathematics courses. Using semi-structured interviews and classroom observations, the study examines how ITAs implement the essential components of CRT, as defined by Gay (2002), while confronting cultural and institutional challenges. Findings revealed that ITAs build inclusive learning communities and negotiate mathematics registers with their students to improve accessibility. The study highlights the pivotal role of culture in teacher professional development, igniting discussion to enhance support for ITAs and promote equity in mathematics education.

Background and Context of the Study

Culture shapes an individual's attitudes, values, and behaviors and, in turn, influences teaching and learning. Teaching is inherently a social and cultural activity, and culturally responsive teaching (CRT) plays a crucial role in fostering inclusive educational environments. CRT involves integrating students' cultural knowledge, experiences, and values into instruction to improve academic and social outcomes (McKinley, 2011; Powell et al., 2016). Theoretical perspectives such as "culturally responsive teaching" (Gay, 2002), "culturally relevant teaching" (Ladson-Billings, 1995), and "culturally appropriate instruction" (Au & Jordan, 1981) emphasize aligning pedagogy with students' cultural contexts to promote equity and meaningful learning.

Cultural mismatches can disrupt the learning process. Hofstede (1986) observed that differences in cultural expectations between teachers and students, such as authority dynamics and communication styles, can lead to misunderstandings. Research underscores transformative potential of CRT to improve student outcomes by connecting instruction to students lived experiences (Aceves & Orosco, 2014; Gay, 2018). Culturally responsive teachers view students' cultural identities as assets rather than obstacles. While shared cultural backgrounds between teachers and students can facilitate this work (Irvin, 1989), cross-cultural contexts present unique challenges, as seen in Florence's (2011) account of an African immigrant teacher whose accent hindered communication despite his efforts to accommodate students.

International graduate teaching assistants (ITAs), who make up a substantial portion of the U.S. higher education instructional workforce, face challenges as they adjust to new cultural and educational contexts (Chellaraj et al., 2008; Kim, 2014). These include balancing academic and teaching responsibilities, navigating communication barriers, and adapting to unfamiliar classroom norms (Jackson, 2020). Although ITAs' cultural backgrounds significantly shape their instructional strategies (Adebayo & Allen, 2020), the practice of CRT in higher education mathematics by ITAs remains underexplored. Differences in prior training, language, and educational norms can affect how ITAs implement CRT (Lee, 2015; Myles et al., 2006).

Mentorship and professional development have proven effective in supporting the cultural adaptation of immigrant and international teachers (Jackson, 2020), but institutional support is often limited. Framing this research within CRT illuminates how ITAs navigate cultural diversity in mathematics classrooms, highlighting both opportunities and barriers to fostering inclusive and equitable learning environments. This study aims to address the research question: *To what extent do international mathematics graduate teaching assistants practice culturally responsive teaching at a U.S. public university?*

Theoretical framework

This study draws on Gay's (2002) CRT framework, which includes five components: cultural diversity knowledge development (CDK), integration of cultural perspectives (ICP) into the curriculum, building inclusive learning communities (BILC), effective communication (EC), and instructional responsiveness (IR). For ITAs, developing cultural knowledge involves understanding students' cultural identities to create equitable classrooms (Gay & Kirkland, 2003; Hofstede, 1986). Integrating cultural perspectives into mathematics helps make abstract concepts more meaningful (Aronson & Laughter, 2016). Building inclusive communities requires fostering a sense of belonging and addressing systemic barriers (Parveen, 2024). Effective communication involves bridging linguistic and cultural differences (Gay, 2018). Instructional responsiveness requires adapting strategies to meet learners' diverse needs (Subban, 2006; Tomlinson, 2015). CRT provides theoretical framework for examining how ITAs in mathematics recognize, value, and integrate students' cultural experiences into their instructional practices.

Methodology

To understand ITAs' teaching practices, I drew on semi-structured interviews that enabled me to explore beyond a predefined set of core questions, ensuring consistency across interviews and enabling insight and context-specific information to be captured in the participants' own words (Creswell & Poth, 2016). Also, through observation, I gathered data on ITAs' interactions, activities, and dynamics in a natural (classroom) setting. A combination of both techniques was necessary for gaining a comprehensive understanding of ITAs' teaching practices.

Six ITAs who lived and trained in Africa and Asia were purposively selected for their distinct cultural identities and experiences with mathematics instruction at a public university in the U.S. Adam (South Asian) was an instructor for Calculus for Technology II, Salley and James (Africans) were instructors for Survey of Calculus, Eve (South Asian) was an instructor for Precalculus, Matt (African) was an instructor, and Kalley (Southwest Asian) was an instructor's assistant, both for Geometry and Measurement for K-8 teachers. All these courses are first-year courses, and the six ITA prior to entering the US had no interaction with Americans throughout their education. Each participant had at least one year of teaching experience at the university; thus, they have all participated in course instructors' meetings for at least a year, discussing contextually practicable and relevant instructional strategies in the American classroom. Ethical approval was obtained from the University's Institutional Review Board before recruitment and participant consent were obtained. The setting for this study is a large public university with over 15,000 students, approximately seven percent of whom are from ethnic minority groups. This indicates that most classrooms are predominantly white, and that an international student serving as a course instructor or assistant in such a class is of particular interest to the researcher. Of even more interest to the researcher is ITA's adaptability to the cross-cultural classroom environment and the practice of CRT.

I developed and pretested an interview schedule and observation guide, which were subsequently refined to align with CRT principles. Classroom observations of each ITA followed individual interviews during the data collection period, all conducted by the researcher. Interviews were audio-recorded, transcribed, and pseudonymized using a researcher-developed coding guide. The six observation notes were similarly processed for analysis.

Data Analysis

A deductive thematic analysis was applied to interview transcripts and observation notes using Gay's (2002) CRT framework. Although all five CRT components guided the study, this paper focuses on cultural diversity knowledge (CDK), building inclusive learning communities (BILC), and instructional responsiveness (IR), the most frequently coded themes. Using NVivo 15, six interview transcripts and six observation notes were cleaned of irrelevant information, read multiple times, and coded sentence by sentence with separate codebooks, which were iteratively refined to align with the CRT framework. Interview and observation data were triangulated to enhance credibility, and transcripts were member-checked before analysis to ensure trustworthiness.

Findings

The analysis of both the interview data and observation notes revealed CDK, BILC, and IR as dominant themes (with a high percentage of codes). Thus, ITAs' self-report and practice of CRT are noticeable and evident in these areas within their mathematics classrooms.

CDK Theme

Cultural awareness, cultural knowledge development, and self-reflection emerged as central codes. All the ITAs expressed sensitivity to cultural diversity in their mathematics classrooms, a commitment to developing knowledge of their students lived experiences and frames of reference, and a continuous practice of self-reflection on their personal biases and cultural orientations. Despite their efforts to build a knowledge base on cultural diversity in the classroom, large class sizes and concerns about potential students' privacy invasion posed challenges for ITAs. Classroom observation revealed that ITAs themselves were key contributors to cultural diversity in their classrooms. With ITAs' awareness of their centrality to classroom diversity, their efforts to overcome cross-cultural barriers to teaching and learning were evident throughout the lesson. For example, one ITA expressed:

> *I don't try to do what I did back at home. So, I try to check in with the instructors' supervisor and the other people who are also teaching this course by asking, how are you talking about this? Because I want to use words they are familiar with and not introduce or say it in a different way to confuse them, I want to build upon what they have already learned in their previous school experience. (Eve)*

BILC Theme

Collaborative learning, inclusion, and promoting learners' sense of belonging were central to ITAs' descriptions of how they facilitate the building of a mathematics community of learners in their classrooms. ITAs mentioned established norms and practices they espouse during mathematics discourse to create safe spaces for students to express themselves freely, and to participate in group activities. Despite these efforts, ITAs reported experiencing difficulties with classroom management due to students' individual differences and preferences. For example, an ITA shared that:

I know how important it is to create a safe environment for students to express their feelings, and no matter what, I let them feel free to express their reasoning in whatever way they feel comfortable. I mean to encourage them, motivate them, and then we can lead their thoughts. (Kayley)

ITAs also emphasized the strategies they adopt and practice to create a conducive classroom environment that provides equal access to all students, irrespective of their cultural identity and entry-level knowledge. For example, Adam shared:

And just in, like study habits and just trying to see that everyone is different, so like also trying to understand what their study habits are, … I want them to walk out with understanding of the content that I'm teaching. And what happens is that I try to do that by taking different methods of teaching. I also use a more pictorial approach where you like to use pictures, add diagrams and pictures that anecdotes so that people can relate.

Such practices include, but are not limited to, group activities, peer assessment, and peer teaching. The classroom observation revealed ITAs' significant interest in encouraging learners to support and accept one another and to help each other succeed academically. I observed instances in which instructors encouraged their students to peer-assess and critique group presentations, thereby fostering value for diverse perspectives. Validation of both group and individual work was a shared responsibility of instructors and students. Most ITAs regularly solicited students' contributions by asking for comments, concerns, clarification, or questions, but seldom walked around to check individual work. For instance, students in two classrooms had a round-table seating arrangement, where they sat in their regular groups and worked together. ITAs were noticed learning about students' entry knowledge during these tasks.

IR Theme

In mathematics instruction, ITAs ensure congruity by using culturally responsive examples that are related to all students, irrespective of their identity. They also highlighted the importance of negotiating instruction approaches with students to incorporate their learning habits/preferences. Five ITAs mentioned using various representations of class activities to accommodate diverse learning styles in their classrooms. Two ITAs, for example, shared:

There were a lot of, like, snow questions and a lot of times it was related to hiking and stuff. I know that a lot of my students go hiking and they go hiking together sometimes … So, the hiking example was when we talked about snow; that thing was very relatable for *them. So, like real things that they have like seen or like they know about and like yeah stuff like that" (Eve); "I also use a more pictorial approach where you like use pictures, add diagrams and pictures that anecdotes so that people can relate." (Adam)*

These instructors believed that context and graphics are valuable means of instruction across cultural borders. Student-centered teaching strategies whereby learners' preferences, experiences, and backgrounds significantly inform classroom discourse received premium attention among ITAs.

Classroom observation further revealed that ITAs utilize a combination of technological tools to enhance students' access to instructional materials and build upon students' relevant prior knowledge to improve communication. Students were actively engaged during mathematics discourses. ITAs consistently drew students' attention and ensured no one was left behind. They complemented their efforts with illustrations and analogies, such as contextual diagram sketches, to support the development of conceptual understanding of mathematics for learners of varying abilities. There were signs of resilience in ITAs' pursuit to overcome their personal biases and cultural inclinations, enabling them to make the necessary connections between the mathematical content and their knowledge of the learners lived experiences. They recounted personal reflections as a key contributor to their pedagogical development.

Discussions and Conclusion

From the results, ITAs expressed and utilized self-reflection in their CDK development and CRT implementation, as well as in reorienting their cultural biases and orientations, aligning with Gay and Kirkland's (2003) recommendation that ITAs must utilize self-reflection and continuous learning to minimize their cultural unfamiliarity. ITAs also appeared to accept responsibility for adapting to the cross-cultural classroom environment, as suggested by Hofstede (1986), which must be required of the instructor rather than the learner.

Building and nurturing a community of mathematics learners is an area where ITAs' beliefs and practices were consistent. The seating arrangement of students in some classrooms highlighted the practice of collaboration and teamwork. ITAs demonstrated a strong commitment to students' academic success by fostering inclusive learning environments and creating safe spaces for learners. Parveen (2024) argues that encouraging collaboration among students across cultural boundaries is an essential requirement for CRT. ITAs, being aware of potential cross-cultural barriers that may hinder learners from sharing any misunderstandings, regularly create opportunities for peer review and validation of learners' thinking and reasoning, and invite learners' questions and comments. This finding is consistent with Gay's (2002) assertion that promoting teamwork and collaboration and ensuring that students feel valued in the learning community are critical to CRT and foster peer relationships across cultural boundaries. ITAs expressed and demonstrated the utilization of culturally responsive examples and tasks, as well as diagrams and illustrations to maximize learners' conceptual understanding of the mathematics content. This practice among participants aligns with the propositions of Subban (2006) and Tomlinson (2015) regarding differentiated instruction in culturally responsive teaching within a culturally diverse classroom.

In conclusion, this study underscores the importance of CRT in mathematics classrooms and the critical role ITAs play in promoting inclusive and equitable learning environments. Despite ongoing challenges, ITAs demonstrate adaptability, commitment to professional growth, and increasing awareness of cultural diversity in their teaching. The findings highlight the need for institutional support—such as CRT-focused professional development, mentoring, and resource structures—to strengthen ITAs' instructional effectiveness. Self-reflection, self-awareness, and responsive listening emerge as essential to ITAs' cross-cultural teaching. A limitation of this study is that, due to space constraints, course-specific challenges to ITAs' CRT practices were not examined in depth; the analysis also focuses more on data from three of the ITAs. Also, this study examines ITAs' self-reported practices and classroom observations without incorporating students' perspectives; future research could explore how students perceive and navigate ITAs' cultural norms and practices to support their learning in mathematics classrooms.

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NUMBER TALKS AS SPACES OF BECOMING FOR PRESERVICE TEACHERS

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This study examined how Number Talks functioned as pedagogical spaces where preservice teachers (PSTs) experienced becoming-teacher navigating uncertainty, disruption, and improvisation. Through analysis of lesson plans, enactment notes, and reflections from 18 PSTs, I demonstrate how Number Talks position PSTs at the intersection of planning and emergence. Drawing on Deleuzoguattarian concepts and Ovens et al. (2016) theorization of becoming-teacher, I argue Number Talks are spaces of becoming where PSTs *learn to inhabit vulnerability, negotiate authority, and recognize disruption as constitutive of teaching*.

Number Talks are brief instructional routines centered on students' mental computation strategies (Parrish, 2010). Students engage in problems through multiple strategies as the teacher records or represents student thinking. Unlike scripted routines, Number Talks demand teachers hold their own mathematical knowledge lightly, follow students thinking, and make instructional decisions in the moment (Parrish & Dominick, 2016). While research highlights benefit for students' number sense (Humphreys & Parker, 2015), little is known about their overall effectiveness (Matney et al., 2020). Number Talks support students' mathematical reasoning while providing PSTs with accessible entry points for developing research-based teaching practices (Raymond & Campbell, 2025; Woods, 2021). Drawing on Deleuze and Guattari's (1987) philosophy, as applied to teacher education by Marble (2012) and Ovens et al. (2016), I examined how PSTs navigate smooth and striated spaces of teaching. Traditional accounts of teacher development have dominated research for decades, often assuming linear progressions that oversimplify the complex work of learning to teach (Raduan & Na, 2020). Following Ovens et al. (2016), I view becoming-teacher not as acquiring stable professional knowledge but as navigating ongoing uncertainty, tension, and contradiction. Within this conceptualization, Number Talks' blend of structure and unpredictability vividly illustrates this process (Marble, 2012; Ovens et al., 2016).

Theoretical Framework

Becoming-Teacher

Becoming-teacher conceptualizes learning to teach as an ongoing process shaped through encounters with uncertainty and classroom complexity (Ovens et al., 2016). This perspective refuses linear development from novice to expert, instead foregrounding how vulnerability and openness animate the process of becoming-teacher. Building on Deleuze and Guattari (1987), Marble (2012) theorized teaching develops through demanding classroom encounters requiring creative responses to new problems. Ovens et al. (2016) argued PSTs occupy a middle space—no longer students of teaching, not yet teachers of students—navigating a nonlinear transition. PSTs must compose themselves alongside student thinking and classroom practice. Vulnerability and uncertainty are not deficits – they are constitutive of responsive teaching.

Smooth and Striated Spaces

Deleuze and Guattari (1987) distinguish between smooth space (open, directional, emergent) and striated space (measured, controlled, structured). Teaching constantly shifts between these models. Lesson planning tends toward striation: teachers define objectives, sequence activities, anticipate responses. Yet enactment is seldom smooth as student thinking flows in unexpected directions, time expands and contracts unpredictably, the boundaries between right and wrong blur. Deleuze and Guattari (1987) argue that these models never appear in isolation but emerge together in practice.

Lines of Flight

Deleuze and Guattari (1987) offer the concept of lines of flight—movements escaping established structures and creating new possibilities. Lines of flight occur when something unexpected happens to deterritorialize the lesson, opening unanticipated trajectories. Student responses deviating from what was planned, errors revealing alternative logic, and questions shifting the focus function as lines of flight. Importantly, lines of flight are generative, forcing PSTs to think differently, to improvise, and to recognize teaching cannot be fully controlled.

Why Number Talks Intensify Becoming

Number Talks intensify becoming-teacher by combining structured anticipation with pedagogical openness, destabilizing authority and positioning student strategies as central content. Student thinking remains unpredictable despite careful planning, with surprises functioning as lines of flight. Finally, Number Talks foreground restraint: teachers must resist showing procedures or immediately correcting errors. These conditions align with Ovens et al. (2016) conception of becoming-teacher, as PSTs navigate uncertainty through challenging encounters rather than predetermined technique.

Methodology

Eighteen elementary PSTs enrolled in a two-course series of methods courses planned, enacted, and reflected on Number Talks. Participants enacted one Number Talk with their peers first semester and enacted the second during field placements (grades K–5). Data included lesson plans (detailing mathematical goals, anticipated strategies, key concepts, and planned questions), enactment notes (documenting what occurred with attention to unexpected responses), and written reflections (addressing surprises, difficulties, wishes, and learnings).

Analysis began with immersion: I read through all lesson plans, enactment notes, and reflections to develop familiarity with data. I took analytic memos documenting emerging patterns. I then engaged in initial coding guided by the research question: *How do Number Talks function as pedagogical spaces where PSTs experience becoming-teacher through navigating uncertainty?* Codes attended to lines of flight, challenges, uncertainty, gaps between planning and enactment, and retrospective recognition of missed opportunities. Following initial coding, I conducted cross-case analysis, comparing patterns across participants to identify themes cutting across individual experiences. This iterative movement between individual cases and emerging themes allowed theoretical concepts to sharpen through repeated engagement with the data. To enhance efficiency in managing the dataset, Claude AI (Anthropic, 2025) was used for organizational support, including formatting, pseudonym assignment, and preliminary coding assistance. Prompts directed Claude to identify textual instances of theoretical concepts (e.g., "locate moments participants describe gaps between planning and enactment"), organize quotes thematically, and flag potential patterns warranting closer researcher attention. All interpretive decisions, theme development, and theoretical connections emerged through researcher's engagement with the data. Coded segments were mapped to theoretical concepts, and cross-case analysis identified three themes cutting across participants.

Findings

Navigating Smooth and Striated Spaces

Participants prepared extensively, yet in every case, enactment surprised them. This collision between planned anticipation and enacted surprise was not incidental—it demonstrated how Number Talks functioned as spaces of becoming. Jordan's plan for 3 × 26 exemplified the thoroughness of preparation. She listed expected strategies: area models, distributive property, repeated addition, and skip counting. She noted key concepts and prepared questions: "How did you break apart the numbers? Why did you choose to start with 20?" This detailed plan created a striated space—a mapped territory where she knew what to expect and how to respond. The students immediately smoothed these striated spaces when they used strategies differently than expected. Rather than deploying area models, they wanted to draw literal arrays.

Similarly, Rowan's carefully scaffolded sequence—23 + 10, 23 + 20, 23 + 40—was interrupted when a student asked, "Why do all of them have 23 in them?" This question—not on Rowan's list of anticipated responses—shifted the entire focus. Instead of solving problems sequentially and discussing strategies, students wanted to talk about structure and design. The Number Talk flowed in a direction Rowan had not mapped.

These collisions, where PSTs experienced disorientation, created the condition for becoming teachers. Parker described encountering a student who "drew 5 groups of 5" when she expected mental decomposition: "I had difficulty representing the strategy on the board because it wasn't something I had prepared for." Parker had to compose herself differently in that moment: not as someone who knows what to do in advance, but as someone learning to think with student reasoning. She later wrote, "I learned to think with their reasoning instead of over it." This retrospective shift from teaching over student reasoning exemplified that becoming-teacher is not about acquiring a technique but recomposing oneself in relation to student thinking.

Crucially, becoming-teacher intensified in retrospection. After the Number Talk, participants recognized what they could have done. Britney wrote: "I wish I had asked… what is another way we can solve this problem?" Finley similarly noted: "I wished I asked why they chose this specific strategy." This retrospective recognition—*I could have done this differently*—is not about inadequacy but about ongoing becoming. As Ovens et al. (2016) argued, becoming-teacher is always provisional, never settled. Teachers continuously discover new possibilities through reflecting on the gap between what happened and what could have happened.

Redistributing Authority

Traditional instruction positions the teacher as mathematical authority: the one who knows the right answer, the efficient method, the proper procedure. Number Talks deliberately redistribute this authority, positioning student reasoning as the mathematical content. For participants, this redistribution was unsettling—it required them to hold their own knowledge lightly. Multiple participants articulated the difficulty of restraining their own mathematical knowledge. Parker wrote: "I struggled not to 'give away a strategy that I would use' and instead used guiding questions." This language – "struggled," "not to give away"—reveals the tension.

Number Talks demanded she hold back and let students build their own understanding. Avery's language was even more embodied, "The toughest part was sitting back and letting the students have their moment to think." The phrase "sitting back" connotes physical restraint. She wanted to step in, to guide, to move students toward efficient strategies. But doing so would have collapsed the space for student authority. Kendall articulated the underlying tension as: "Difficult to balance the right way to solve and representing students' thinking." This revealed a fundamental uncertainty about authority: Is there a "right way"? If so, should the teacher guide students toward it? Or should all student strategies be positioned as equally valid?

The redistribution of authority was most visible when students actively resisted PSTs goals for the Number Talk. Blake planned an "adding-up" approach for subtraction—starting from the smaller number and counting up to the larger. Yet students "insisted on stacking subtraction instead of adding-up strategy." They refused to adopt her method. Blake reflected: "Most difficult part was to get them to break away from what they are used to doing." Her language, "get them to break away", revealed the tension. She wanted to redistribute authority to students, but only if they used the strategies she valued. Blake experienced students' insistence on familiar methods as resistance rather than agency. Similarly, Casey's students "wanted to keep using a place value chart, even for small-number tens problems." Casey had planned for mental mathematics, but students preferred visual, structured representations. This preference—a line of flight—forced Casey to decide: Do I honor student thinking when it diverges from my goals? Or do I redirect them toward what I believe is more appropriate?

Some PSTs repositioned themselves in relation to student authority. Cameron's student solved triangle area by calculating "6 × 5 = 30, then halved it". Rather than redirecting toward formula (A = ½bh), Cameron followed the student's logic: "I asked him to explain why halving made sense, and he showed me how two triangles make a rectangle." In this moment, Cameron composed himself—not as the one who knows the formula and evaluates student approximations of it, but as someone learning alongside the student about how area relationships work. This demonstrated becoming-teacher is shaped by allowing student thinking to surprise and instruct.

Navigating Chaos and Affect

PSTs' reflections were saturated with affective language: struggling, wishing, difficulty, challenge, chaos. Traditional frameworks might position these emotions as transitional discomforts diminishing with experience. Yet Ovens and colleagues' (2016) framework suggested affect and uncertainty are not byproducts of being novice but constitutive features of teaching itself. Number Talks, by generating unpredictability and requiring improvisation, made this affective dimension visible and intense.

Participants used the word “chaos” or “chaotic” to describe their Number Talks. Alex wrote about “helping the students stay focused within a chaotic environment.” Emerson noted “students kept talking over one another” and found it “difficult for me to have them share their process.” This chaos was not simply about classroom management; it was about the unpredictability inherent to centering student thinking. When the teacher controls discourse tightly, classroom interaction becomes more orderly. Number Talks loosen this control when students generate multiple strategies, share simultaneously, or interrupt to add on or disagree. The discourse becomes polyvocal and harder to orchestrate. For participants, this felt like losing control, yet it is this loosening that makes Number Talks mathematically productive.

Several participants described difficulty with logistics. Cameron noted: “Most difficult... was managing the pacing.” Time in Number Talks does not follow linear progression, and some strategies took longer to explain than anticipated. This temporal unpredictability characterizes smooth space. Rowan also wrote: “The most challenging aspect was ensuring all students had opportunities to share while managing time.” Striated time is measured and controlled: five minutes for introduction, ten minutes for sharing, two minutes for closing. Smooth time expanded and contracted based on what emerged, disorienting participants but reflecting the genuine cognitive work of mathematical thinking. PSTs also navigated student uncertainty and frustration. Riley described “managing students’ confidence levels” when she posed 496 ÷ 4 and students “immediately shut it down”—they refused to attempt it, overwhelmed by the numbers. She broke it down to 400 ÷ 4 first, then added 96 ÷ 4. She wrote: “Once they saw they could do part of it, they re-entered.” Her improvisation involved both mathematical and emotional labor—managing the affective space of the classroom, not simply delivering content.

Some PSTs articulated feeling inadequate. Parker could not represent a student’s visual strategy. Blake could not persuade students to use her preferred method. Rowan’s careful sequence was disrupted. These moments could be read as failure. The teacher did not have enough knowledge, authority, and skill; yet these moments are generative. They reveal teaching as ongoing negotiation rather than mastery. As Ovens et al. (2016) argued in their theorization of PSTs occupying middle spaces, becoming involves tension and instability. Teachers do not overcome vulnerability by acquiring more knowledge; they learn to inhabit vulnerability as a condition of teaching responsively (Marble, 2012).

Number Talks intensified affective experiences because they required improvisation and holding space for student thinking even when it generated chaos. This affective intensity made becoming-teacher visible: participants experienced themselves differently in these moments—uncertain, vulnerable, not-in-control—and had to compose new relationships to authority, knowledge, and students. Teacher education often attempts to minimize preservice teachers' anxiety by providing scripts and procedures. Yet Number Talks do the opposite: they deliberately generate uncertainty as a productive pedagogical condition.

Discussion

Drawing on Ovens et al. (2016) conceptualization of becoming-teacher, this study positions Number Talks as pedagogical architectures deliberately generating uncertainty, redistributing authority, and foregrounding reflection. These three features address the central question of the study. First, the demand for anticipation without predetermination: participants had to plan extensively yet remain radically open to what students did. This created productive tension—participants could not plan themselves out of uncertainty but had to prepare thoroughly and hold those preparations lightly. Second, the redistribution of mathematical authority: the routine structurally positioned student strategies as the mathematical content. PSTs had to record student thinking, facilitate discussion among strategies, and resist immediately evaluating correctness, forcing them to reposition themselves and trust student thinking even when it diverged from expectations. Third, the tension between intention and enactment created a space where participants recognized alternative possibilities.

Traditional frameworks position unexpected student responses as disruptions to be managed, treating them as evidence of inadequate planning or student confusion (Raduan & Na, 2020). Yet these moments functioned as lines of flight—invitations to think and teach differently. When Rowan's students noticed her embedded pattern, when Avery's students used relational reasoning, when Cameron's student decomposed triangle area creatively—these were pedagogical gifts opening unforeseen trajectories.

Conclusion

In these moments—when Avery sat back to let students "have their moment," when Parker resisted giving away her strategy, when Riley scaffolded students back from shutdown, when

Rowan's careful sequence was interrupted by student noticing—PSTs were becoming-teachers as they were learning to inhabit vulnerability, to trust student thinking, to improvise in response to surprise, to recognize retrospectively what they had not seen in the moment. Number Talks, with their structural demand for anticipation without predetermination, their redistribution of mathematical authority, and their generation of retrospective gaps, functioned as spaces where becoming-teacher happened intensely and repeatedly. They are not routines to be mastered but pedagogical environments continuously reshaping preservice teachers through disruption and improvisation. This study contributes to understanding learning to teach mathematics as ongoing becoming—not a developmental progression toward mastery but a continuous negotiation of teaching responsively.

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TRANSFORMING CALCULUS EDUCATION TO ENHANCE SELF-EFFICACY FOR DIVERSE LEARNERS

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This study investigates student self-efficacy in a first-semester calculus course redesigned with innovative, student-centered pedagogies to support underrepresented minority (URM), first-generation, and low-income students. Observation, survey and interview data were collected to assess the impact of these distinct pedagogical approaches on students' non-cognitive achievements, focusing on the role of cooperative learning in fostering students' confidence and self-efficacy. Our findings reveal significant differences in students' non-cognitive achievements between traditional and reformed calculus instruction, highlighting the potential benefits of reformed teaching practices in promoting URM students' confidence, self-efficacy, and overall success/retention in STEM fields.

Calculus is widely recognized as a gateway course for students pursuing degrees in STEM fields (Bressoud et al., 2015). However, it is also notorious for its high failure and dropout rates, particularly among students from underrepresented minority (URM) groups (Koch & Drake, 2018). Typical Calculus courses primarily address the question of *how* to do calculations and are delivered as a lecture with minimal interactivity with students (Epstein, 2013). The questions of *why* these calculations are needed and *what* the problems that such calculations can help to solve receive significantly less attention. Such an approach to Calculus instruction often leaves students unmotivated and disengaged since they lack an understanding of the need for these mathematical tools and do not spend sufficient time learning how to solve practical problems with the tools of Calculus (Rasmussen & Ellis, 2013). These challenges contribute to the persistence of achievement gaps and hinder the success and retention of diverse student populations in STEM disciplines (Estrada et al., 2016).

There is now widespread support for incorporating "active learning" methods in advanced mathematics and science courses. This approach emphasizes tackling challenging problems through collaborative efforts. Previous research on cooperative learning in undergraduate mathematics has supported the use of small group work and provided guidance on its implementation (Ahmadi, 2002; Reinholz, 2018; Rogers et al., 2001). However, there has been limited research on how these methods impact students' self-efficacy, especially among underrepresented minority students. Recognizing the importance that both active learning and self-efficacy play in the learning of calculus and students' retention in STEM, we aim to explore underrepresented minority students' self-efficacy in a reformed calculus course. Through this study, we want to address the following research questions: RQ1: Is there a significant difference in students' self-efficacy for URM students in reformed and traditional calculus classes? RQ2: How are these differences moderated by key demographic characteristics such as gender? RQ3: In which ways do the reformed calculus course help foster URM students' self-efficacy?

Theoretical Framework

This study draws on Social Constructivism and Self-Efficacy Theory to examine how collaborative learning and active engagement strategies impact URM students' confidence and learning in calculus. Social Constructivism emphasizes that learning is a social process, where knowledge is constructed through cultural and social interactions (Vygotsky, 1978). The concept of the Zone of Proximal Development (ZPD) highlights the importance of collaborative environments that enable students to engage in tasks with guidance, promoting deeper understanding through dialogue and shared problem-solving (Palincsar, 1998). This cooperative learning approach fosters a sense of community and belonging, which is particularly beneficial for URM students.

Complementing this, Bandura's (1997) Self-Efficacy Theory focuses on individuals' beliefs in their capabilities to succeed in specific tasks. It identifies four sources of self-efficacy: mastery experiences, vicarious experiences, social persuasions, and emotional states, all of which influence students' confidence and persistence. Strengthening self-efficacy is crucial for URM students in overcoming challenges and barriers in calculus, fostering resilience and engagement (Lent et al., 2000). By integrating these frameworks, this study examines how collaborative and active learning strategies within the calculus curriculum can support URM students' self-efficacy and engagement, offering a comprehensive lens for understanding the impact of pedagogical innovations.

The conceptual framework for this study centers on how the integration of cooperative learning, real-world applications, and formative assessments—key elements of active learning—enhances URM students' self-efficacy and performance in calculus. Cooperative learning encourages collaborative problem-solving and peer interactions, allowing students to construct knowledge collectively and build confidence through shared experiences (Johnson et al., 1991). This interaction aligns with social constructivist principles and offers URM students' opportunities to engage with complex calculus concepts, reinforcing their self-efficacy through vicarious learning and positive social feedback. Real-world applications, on the other hand, involve incorporating authentic examples that connect calculus to practical scenarios, making abstract concepts more relatable and engaging for students (Prince, 2004). This approach helps URM students recognize the relevance of calculus in various contexts, thereby enhancing their belief in their capabilities. Lastly, formative assessments provide ongoing feedback that enables students to monitor their progress, recognize achievements, and identify areas for improvement (Black & Wiliam, 1998). This iterative process supports mastery experiences, thereby reinforcing self-efficacy and fostering a growth mindset. Together, these active learning strategies create an inclusive environment that supports URM students' confidence and success in calculus. This study explores how these elements interact to enhance self-efficacy and academic performance, providing insights into effective pedagogical practices in STEM education.

Method

To address persistent equity gaps, a calculus course at a West Coast minority-serving institution was redesigned from a traditional lecture-based model to a student-centered model emphasizing cooperative learning and self-efficacy (Chavez, 2023). This reformed pedagogy was implemented in distinct course sections, which were compared to sections retaining the traditional format during the same academic semester. The traditional and reformed sections shared the same learning objectives, textbook, primary assessments (exams), and total instructional time (50-minute sessions, four times per week over a 15-week semester). Class sizes and student demographics (e.g., year in program, major distribution) were also similar. The sole planned difference was the pedagogical approach used during class sessions.

The reformed course implemented a suite of pedagogical innovations designed to foster active engagement, collaboration, and self-efficacy: cooperative learning in small groups (i.e. collectively solving complex calculus problems and employing peer instruction to explain concepts), integration of real-world applications (the curriculum incorporated contextual problems from fields such as economics, biology, and engineering), formative assessment and responsive feedback (emphasized continuous low-stakes assessment).

A mixed-methods approach was employed, combining both quantitative and qualitative data collection and analysis methods. Observations were conducted to document and contrast instructional practices and student interactions. The reformed pedagogy, as described above, was applied to every class session in the reformed section throughout the semester. From the total of

60 course sessions (4 sessions/week x 15 weeks), a purposive sample of 22 sessions was selected for structured observation: 11 from the traditional section and 11 from the reformed section. Observations were stratified across the semester (weeks 3-14) to capture a variety of topics and avoid atypical introductory and review sessions. Observations focused on the nature and frequency of student participation, question types, and classroom dynamics. Detailed field notes were supplemented by audio recordings to ensure accuracy.

A survey ($N = 139$) was administered to collect quantitative data on students' non-cognitive outcomes. It comprised Likert-scale questions measuring students' confidence in their mathematical abilities, comfort in classroom participation, and overall motivation. Demographic information, including gender, first-generation status, and socioeconomic status, was also collected. To provide deeper understanding of students' responses in the survey, semi-structured interviews ($N = 12$) were conducted with a subset of URM students from both the traditional and reformed calculus courses. The interviews aimed to gather in-depth qualitative data on URM students' experiences, perceptions of the instructional methods, and the specific ways in which these methods influenced their confidence and engagement. Each interview lasted approximately 15 minutes and was audio-recorded for subsequent transcription and analysis.

Independent samples t-tests were conducted to compare non-cognitive outcomes between different groups of students (e.g., traditional vs. reformed classes, male vs. female students) for survey data. The dependent sample t-test was used to see the relationships between students' demographic characteristics and their reported non-cognitive outcomes. Classroom observations, student survey open-response items and interviews were analyzed separately using open and selective coding and frequency counts. Bandura's (1997) four sources of self-efficacy (mastery experiences, vicarious experiences, social persuasions, and emotional states) and key elements of active learning (cooperative learning, real-world applications, and formative assessments) was used for initial coding. Later, new categories were added to the coding scheme as analysis progressed. Each researcher uses these qualitative data sources, and the two researchers meet to discuss the categories to maintain consistency of code creation. A final round of selective coding led to categorizing the frequency of students' questions, group work time, observed patterns of instruction and interaction, and student responses. Although there was overlap between coding themes and categories amongst these three qualitative data sources, each set of qualitative data was coded separately and then used to triangulate the findings.

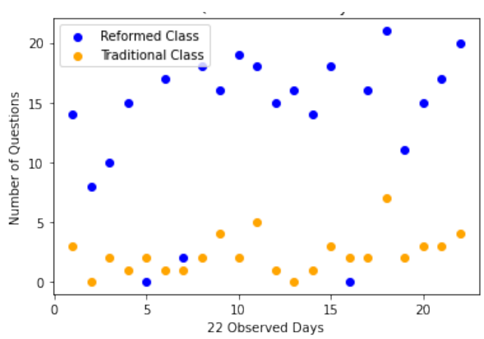
Findings

Comparative analysis of traditional and reformed calculus courses highlighted significant differences in non-cognitive outcomes. Students enrolled in the reformed math classes reported higher levels of various non-cognitive outcomes than those in traditional classes. For instance, students in reformed classes felt more comfortable asking questions in class ($M = 3.21$, $SD = 1.25$) compared to those in traditional classes ($M = 3.57$, $SD = 1.21$), with a significant difference noted ($t(140) = 1.74$, $p = .04$). Additionally, reformed class students reported greater comfort in working collaboratively with others on math tasks ($M = 3.71$, $SD = .96$) compared to their traditional class peers ($M = 4.04$, $SD = 1.12$), reflecting a significant difference ($t(140) = 1.868$, $p = .03$). These results suggest that the reformed pedagogy positively influences student engagement and comfort in collaborative learning environments.

Figure 1 compares the number of questions asked by students in the reformed and traditional classes over the same observation period. The data reveals that, on most observation days, students in the reformed class asked significantly more questions ($M = 13.36$, $SD = 6.18$) than those in the traditional class ($M = 2.32$, $SD = 1.64$), $t(21) = 7.65$, $p < 0.001$. This finding suggests that the reformed class fostered a much higher level of student engagement and interaction during lectures compared to the traditional class.

Figure 1

Number of Questions Asked by Students for Each Observation Day



A key finding was the transformative role of collaborative learning in fostering engagement and mitigating participation barriers for underrepresented minority (URM) students. Students described a structured group-work model—where they grappled with problems in small, randomized groups before whole-class discussion—as pivotal to their experience. This approach directly addressed initial feelings of intimidation and self-doubt; one student (Student B) recalled a prior failure in calculus that made her question her abilities, while another (Student T) admitted to being "super nervous" and "scared to even talk." The collaborative environment alleviated these concerns by normalizing the need for help. Students reported that randomized groups created a low stake setting where it was "easier to ask for help," as they could see others also struggling and could rely on peers who "understood the concepts very well" to provide explanations. Furthermore, this structure facilitated social connections, with Student T noting that being "pressured to talk to our groupmates" helped him become "more comfortable" and even make friends. Ultimately, the cooperative format transformed their relationship with seeking help, reframing it from a solitary, intimidating act to a collective endeavor, as Student T summarized: "It was easier to seek for help if you're together seeking for help." This suggests that intentionally structured group work can be a powerful lever for creating inclusive and supportive mathematics learning environments for URM students.

Interestingly, the survey data revealed notable gender differences in students' non-cognitive outcomes within the calculus courses. Male students reported significantly higher levels of comfort in various classroom activities compared to their female counterparts. Specifically, male students expressed greater ease in asking questions during class ($M = 3.62$, $SD = 1.12$) than female students, who reported lower comfort levels ($t(139) = 2.50$, $p = .01$). Additionally, male students were more likely to contribute their ideas and suggestions during classroom discussions ($M = 3.52$, $SD = 1.16$) compared to female students ($t(139) = 1.845$, $p = .03$). The openness to discussing questions with classmates or instructors was also significantly higher among male students ($M = 3.98$, $SD = .84$) than female students ($t(139) = 3.01$, $p = .002$). Furthermore, male students reported a stronger preference for group work in math class ($M = 3.75$, $SD = 1.11$) than their female counterparts ($t(139) = 2.31$, $p = .01$). These findings indicate a gender disparity in classroom engagement and comfort, suggesting a need for targeted interventions to support female students in calculus courses.

Conclusion

This study demonstrates that a reformed, student-centered calculus pedagogy significantly shapes the classroom environment and students' psychosocial experiences, with important differential impacts across demographic groups. Our findings confirm that focusing on both cognitive and non-cognitive factors is crucial for creating equitable STEM pathways (Rasmussen & Ellis, 2013). Specifically, the reformed course fostered self-efficacy among URM students through key mechanisms such as cooperative learning, exploratory tasks, and a supportive classroom climate (Bandura, 1997; Johnson et al., 2014). These features directly addressed Bandura's sources of self-efficacy, enhancing students' confidence, participation, and persistence. Importantly, the integrated analysis revealed that the benefits and experiences of the reformed pedagogy were not uniform. While the intervention supported URM students broadly (Estrada et al., 2016), a persistent gender disparity in classroom comfort was evident. Survey data revealed that male students reported significantly higher comfort levels than female students in key interactive aspects of the course, including asking questions in class, contributing to discussions, and preferring group work (Eddy et al., 2014; Laser et al., 2021). This indicates that pedagogical reforms can simultaneously nurture inclusivity for some groups while unintentionally leaving gendered patterns of participation unaddressed, underscoring that equity-oriented design must be attentive to intersecting identities (Harper, 2010).

For practitioners, these results argue for the widespread adoption of interactive, student-centered pedagogies in gateway STEM courses, as they create more inclusive conditions for success, particularly for URM students. However, instructors should implement these methods with explicit attention to group dynamics, participation patterns, and nuanced feedback to ensure all students—across gender and racial/ethnic lines—can fully benefit. Future research should build on these findings in two key directions. First, longitudinal studies are needed to determine whether the observed gains in self-efficacy translate into greater persistence and success in subsequent STEM courses and careers. Second, to advance equitable design, research must employ intersectional frameworks to investigate *how* and *why* students of different genders and racial/ethnic backgrounds experience reformed pedagogies differently. Such work will be essential for developing more targeted, effective strategies that support every student in realizing their potential in STEM.

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Tools, Technologies, and Representational Supports for Mathematical Learning

THE UNEVEN INFLUENCE OF MATH APPS: FINDINGS FROM AN EXPLORATORY MULTIPLE-CASE STUDY

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Math apps are becoming increasingly common, with most research examining achievement-related outcomes. Using an exploratory multiple-case study methodology, I examined the relationships between five third graders' math app use and their mathematical identities and motivation. I found that students felt math apps were generally helpful for learning math (e.g., Reflex's Fact Family Pyramid feature). Some students were also motivated by the multiplayer battling of apps and the environment of playing math apps. The findings suggest specific features of math apps directly relate to students' mathematical identities and motivation, and that students have positive and negative experiences with math apps.

In the years following the COVID pandemic, math apps have become increasingly present in K-12 schools across America and even the world. With growing class sizes, a shortage of teachers, limited funding, and a continued need to make up COVID learning losses (Darling-Hammond et al., 2023), the use of math apps has continued to proliferate in K-12 schools. In fact, Tafradzhiyski (2025) estimates a 129% increase in the number of educational app users from 2018 to 2023. With math apps, like Zearn, accounting for "1 in 4 elementary students nationwide" (Zearn, n.d.), respectively, it is clear that math apps have become a regular part of K-12 education. Given math identity and motivation are important mediators of students success, (Radišić et al., 2024), in this study, I share the trends and patterns of five third-grade students' math identities and motivation over the course of a 5-month-long semester of regular math app use by answering the following questions: (1) What is the relationship between third-graders' math app use and their mathematics identities and motivation in mathematics? (2) What features of math apps relate to students' mathematics identities and motivation in mathematics?

Theoretical Framework & Related Research

Research on math apps has broadly fallen into one of two categories: research on achievement-related outcomes and research on the design and content of math apps. Most research on math apps has centered on achievement (Outhwaite et al., 2022), and results have shown that math apps generally improve achievement (Kim et al., 2021). However, math apps' influence on student achievement depends on how learning is measured. Kim et al. (2021) found that when learning was measured with a researcher-designed assessment, math learning gains were significantly higher than when learning was measured with standardized outcomes (e.g., state math exam). While research has analyzed the design and content of math apps from different aspects, the results have been largely the same: math apps lack mathematical richness and were drill-and-practice based (Cayton-Hodges et al., 2015).

Students' math identity and motivation are important components of their well-being and learning that have been shown to impact their achievement in math (Radišić et al., 2024). Math app research related to elementary students' motivation has shown that math apps generally motivate students (Fadda et al., 2022), but little research has been carried out to show *what features* of math apps and *why* math apps are motivating (or not) for students. And no recent research exists on the link between elementary students' math app use and their math identities.

Using Cribbs et al. (2015) framework, I view math identity as made up of three interrelated components: a student's view of math (Perspective of Math (PM)), a student's view of themselves related to math (Introspection (Intro)), and a student's view of how they are recognized in math (Recognition (Rec)). Each component of math identity had a parallel component related to math app math, or math done on math apps (PM_T, Intro_T, Rec_T). For motivation, I utilized the basic needs from self-determination theory (Deci & Ryan, 1985), which state that out of a desire to fulfill the need of autonomy (A), competence (C), and relatedness (R), people are motivated. I also added the interest (I) task value from expectancy-value theory (Wigfield & Eccles, 2000), as my pilot indicated the enjoyment of math apps was particularly motivating and a component of motivation not accounted for by self-determination theory. Each component of motivation also had a parallel motivation component related to math app math. Table 1 defines my operationalization of each math identity and motivation component.

Methods

Setting, Participants, and Data

This study had five third-grade students from the same classroom from a public elementary school located in the southern United States. These students engaged with math apps as part of their mathematics learning. This study utilized an exploratory case study methodology, with each student as an individual case, as it sought to understand and articulate the relationship between math app use and students' mathematical motivation and identity. I used extensive, multiple sources of data to provide a detailed, in-depth picture of each student. This included weekly observations, weekly short surveys, longer surveys, and interviews. In all, over 75 hours were spent observing the classroom and describing the environment over the span of data collection.

Table 1

Components of Math Identity (MI) & Motivation (MOT) with Definitions

MI & MOT Components	Definition
PM/PM_T	A student's view of math/of math in math apps.
Intro/Intro_T	Students' beliefs about their ability to understand and perform in math/on math apps.
Rec/Rec_T	Students' perception of how others view them in relation to math/math app use.
A/A_T	The desire or ability to control one's math/math app learning.
C/C_T	The perceived ability to master and achieve a math assignment or task (on a math app).
R/R_T	The desire to connect with others and experience a sense of belonging in math/on math apps.
I/I_T	The inherent, immediate enjoyment one gets from engaging in a math activity or task (on a math app).

The Math Apps

In the third-grade classroom, three math apps, Reflex, Boddle, and Prodigy, accounted for the majority of math app time, and math apps were used on average two hours a week. In this section, I will describe these three apps (see Table 2). Reflex was an app focused on fact fluency, and the third graders started out with addition and subtraction facts and were moved to multiplication and division facts halfway through the year. Reflex offered several games that students could play to practice and build math fact fluency. Boddle and Prodigy were similar in nature, and both allowed students to freely roam a game world and choose other online students to battle. In a battle on Boddle or Prodigy, students had to answer math questions correctly to have the chance to attack their opponent. Winning battles allowed students to earn various awards, such as game tokens and coins, which could then be used to buy clothing items for their game characters. Using Outhwaite et al.'s (2022) classification, Reflex would be classified as a practice-based app, as its focus is to help students acquire math skills and knowledge. Boddle and Prodigy have the focus as Reflex, but the "learning content is embedded within a broader immersive player narrative" (p. 8), making them game-based apps.

Table 2

Summary of Math Apps and Key Features

App	Practice-Based App	Game-Based App	Feature Present			
			Multiplayer	Leaderboard/ ranking	Choice of game path/ games	Customization & rewards
Reflex	✓				✓	✓
Boddle		✓	✓	✓	✓	✓
Prodigy		✓	✓		✓	✓

Analysis

To analyze the interviews, a priori coding was applied using the six codes for math identity (PM, PM_T, Intro, Intro_T, etc.) and eight codes for motivation (A, A_T, C, C_T, etc.) found in Table 1. Using thematic analysis (Braun & Clarke, 2012), subcodes were created to explain each case's math identity and motivation. After developing a subcodebook, coding reliability was assessed by calculating the agreement between the primary coder and two independent experts, one in math identity and one in motivation. A subset of 30% of the dataset was randomly chosen; following detailed discussions on the subcodebook and training on 15% of the data, both secondary coders and the primary coder independently coded 15% of the data, achieving interrater agreements of 99% for both sub-codebooks (math identity and motivation), and Cohen's Kappa values of $\kappa = 0.60$ and $\kappa = 0.66$ (math identity and motivation). The weekly surveys and observational notes were organized into tables that kept track of students' responses over time and were analyzed using thematic analysis to identify specific patterns and trends.

Findings

In this section, the similarities and differences among the five cases, Alexander, Madelyn, Nick, Patrick, and Sarah, are shared by identifying patterns and themes that relate the cases to one another with respect to their math identities, motivation, and math app use.

Theme 1: Cycles with Competence & Effort

One pattern that emerged among several of the cases is a cycle that involved effort, success, competence, and sometimes enjoyment. Sarah described this cycle when she said, "If you feel confident, then you're probably going to try harder. Then you'll get it right, and you'll feel, and you'll still be confident, and then it goes in a circle." For Sarah, effort on math assignments was often directly related to her sense of competence. Sarah's competence seemed to support her effort and, eventually, success in solving math problems.

Unlike Sarah, Nick's experience of competence, effort, and enjoyment related to his math app use. Nick said, "It [math apps] makes you learn more math, and then the more you do it, the more you'll get better at it." Nick believed that math apps helped him learn math, which led to a desire to put forth more effort (i.e., "the more you do it"), followed by the experience of getting better and experiencing competence. He also described an enjoyment that came from feeling smart and being successful in math. Nick said, "I like math because I get to learn to be smart."

Madelyn described a cycle involving competence and success. Madelyn tended to "just do the algorithm easily" and follow steps to answer problems and facts, which led her to feel positive about her ability and competent when she fulfilled what it meant to do math for her. For Madelyn, doing math meant solving math facts and answering questions correctly. She said, "If I'm doing good, like if I have most of my facts right, it makes me feel good." One math app Madelyn found to be particularly helpful for learning math was Reflex because "there's a bunch of games on it [Reflex], but it [also] actually helps you learn. So that one's fun and it helps you learn." Madelyn found the Fact Family Pyramid on Reflex helpful for learning math "because then I know what I need to work on, and I know what I'm fluent on." As students became fluent with math facts, each family circle would change color from white to green. Madelyn found this feature to be beneficial in becoming competent and achieving success in math.

Theme 2: Finding Value in Challenging Math

Another commonality among cases was the value students found in challenging math. Sarah described an enjoyment of doing hard math when she said, "I like hard math problems." For Sarah, there was enjoyment in tackling hard math problems and putting in hard work ("I kind of like, I like work") because it helped her grow in a way that doing math she already knew did not allow. Sarah said, "If you only do the math facts you know that you would choose, you're never going to learn anything." She valued hard math because of the opportunities for learning that were absent when working on math that was familiar and known.

Similarly, Nick found value in doing challenging math. Nick enjoyed hard math, saying he "like[d] to face my fears." Much of Nick's value in doing hard math came from the mindset he equipped himself with when doing math. Describing his mindset, Nick said, "If you think positive, then maybe you can get the question right." This belief in himself and his ability allowed him to approach tough problems with a positive attitude. Patrick said he enjoyed hard regular math "because you got to see if you're good at it or not. Or else you'll never be good."

Patrick felt there was some value in doing hard, regular math—math he might not be good at—because it would help him figure out his skill level and get better at math. Patrick also described finding value in hard math app math and said, "If you do harder math, the more math you will know. Because if you choose easy math, you won't know other questions. And you don't know how to solve it." Patrick felt that the more challenging math app math allowed him to grow as a learner, as seen in his ability to answer more questions.

Theme 3: Valuing Minimal Autonomy

A pattern related to some students' positive views of hard math was the valuing of minimal autonomy. Sarah placed importance on not having the freedom to completely control her learning because she felt it was important to be challenged and believed growth came from being assigned problems as opposed to choosing problems for herself. She shared, "If you only do the math facts you know that you would choose, you're never going to learn anything, and a teacher will challenge you."

Nick also valued not having autonomy, saying, "You're not supposed to have that [autonomy] because if we do [have] freedom, then our teacher will get mad at us." Nick felt that he and his classmates could not be trusted with autonomy since "if we could take over math, everyone would just try to get out their computers and do math on there instead." Nick saw value in external rules that limited activities that might not be in his best interest, saying, "because sometimes if you're like free, you would want to play on your iPad all day, but you got to be careful when you have the freedom for the day." Nick's value of not having autonomy stemmed from not trusting himself to stay on task, as he felt tempted to play math apps on his laptop.

Patrick also described a minimal experience of autonomy that he valued. He felt he would rather "be told what to do because if I finish it earlier, I can do more stuff." When Patrick was directed to what he needed to complete, he had an easier time finishing an assignment. Then, he could do other things like math apps or independent work, which he preferred over regular math. While these three students valued not having autonomy for different reasons, all three students found value in not having freedom and choice over their math learning.

Theme 4: Relatedness from Math Apps and the Math App Environment

An interesting theme that emerged among a few participants was the enjoyment of the environment in which math app engagement took place. Generally, math apps were used during free time, where students could choose to use math apps among several options. Free time often meant students could work wherever they wanted around the classroom, whether that be lying on their stomachs on the carpet next to a friend or sitting on bean bags in the corner of the room. A couple of students articulated enjoyment that came from connecting with peers during math app time. To be clear, this connection was not a result of utilizing the multiplayer or battling features that did exist on math apps; rather, the connection came from talking and chit-chatting in the environment that playing math apps provided.

Madelyn described experiencing greater relatedness with math apps because during math app time, “We could talk, or we usually lay down over playing Boddle next to each other and chit-chat.” For Madelyn, the environment of being on math apps provided relatedness and time to talk and connect with her friends. Alexander recounted a similar experience. He commented, “You can connect online, you can have two computers next to each other, playing on different devices, [you can] talk in the game, or physically.” It wasn’t just that math apps allowed interaction *within* a game; Alexander felt the math app environment also offered a connection with classmates. Sarah also described math apps as offering more interaction with peers than regular math because students usually talked more with each other during math app time.

Discussion and Implications

Using a qualitative lens to explore nuanced relationships between third graders’ use of math apps and their math identities and motivation, I expanded the field’s understanding of how math apps relate to the experiences of elementary students. I found that several students were motivated to put forth effort to experience success, and, for students like Nick and Madelyn, math apps were beneficial for learning math. Features like Reflex’s Fact Family Pyramid served as an indicator for Madelyn’s ongoing competence and mastery of math facts. I also found that several valued limited autonomy as they struggled with the desire to be on their math apps during other instructional time. This was an unexpected finding that is contrary to much of the research on autonomy and children. Students also found relatedness not only in the multiplayer features of math apps, like battling with other students on math apps, but also in the environment in which math apps were often utilized, and sometimes more relatedness in the latter. In closing, over thirty years later, I believe the words of Louis Gerstner, “Computers are magnificent tools for the realization of our dreams, but no machine can replace the human spark of compassion, love, and understanding" (Griffing, 1997), still ring true and are echoed by Sarah and several classmates who enjoyed face-to-face connection over virtual math app connection.

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RESPONSE PROCESS VALIDITY EVIDENCE FOR THE PROBLEM SOLVING MEASURE – COMPUTER ADAPTIVE TEST (PSM-CAT)

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The purpose of this proceeding is to round out a validation argument for a computer adaptive mathematical problem-solving test designed for grades 6-8 called the Problem Solving Measure - Computer Adaptive Test (PSM-CAT). This paper synthesizes past validation claims into one publication and adds an additional focus on response process validity evidence. Findings from this response process study indicate that the PSM-CAT has claims related to all validity sources described in the Standards (AERA et al., 2014).

Introduction

Mathematical problem solving has been and continues to be a focus for supporting broad STEM education (National Science Board, National Science Foundation, 2021). Problem solving is a key factor in mathematics learning standards (CCSSI, 2010) and classroom instruction (Li & Schoenfeld, 2019). If problem solving is a part of instructional standards, then logically it should be assessed in a way that leads to valid results and conclusions that students, teachers, and others can use with confidence. The quality of a test hinges on its results and interpretations, which is tied to validity (AERA et al., 2014; Kane, 2013). Validity is a central feature that must be examined for any test because the quality and quantity of validity evidence supporting claims about a test directly impact on a test's results and interpretations (AERA et al., 2014). Administering a classroom-based test lacking strong validity evidence may lead to detrimental outcomes (Bostic, 2023), which suggests the importance of thoroughly examining validity for classroom-based tests. This proceeding provides evidence and argument related to the Problem-Solving Measures-Computer Adaptive Test (PSM-CAT) for K-12 educators and researchers.

Relevant Literature

We draw on three frameworks for the PSM-CAT. A mathematical problem is grounded in Schoenfeld's (2011) framework as a task that cannot be solved at first glance (i.e., complex), the number of solutions is uncertain, and the pathway to the solution is unknown (i.e., open). Verschaffel and colleagues (1999) articulate a framework for a mathematical word problem such that the task must be complex, open, and realistic. The realistic piece complement Schoenfeld's framework with the idea that a word problem must be real or believably real to a problem solver. Finally, we ground the PSM-CAT in Lesh and Zawojewski's (2007) framework for mathematical problem solving, which includes critical thinking, reflecting, and revising ideas to reach a desirable outcome to a problem. Situating the PSM-CAT in established frameworks supports and extends the deep scholarship associated with mathematical word problem solving (Liljedahl & Cai, 2021). Tests must be available that produce valid results and interpretations to further advance mathematics education scholarship and classroom learning. Additionally, aligning to technological advances like computer adaptive testing is desirable.

A core idea of computer adaptive testing (CAT) is to effectively measure a person's outcomes precisely and reliably with as few items as possible (Davey, 2011; Weiss, 1982). It typically begins with showing an individual an item of moderate difficulty. If an individual answers it correctly, then they are given a more difficult item. If they answer it incorrectly, then they are shown a less difficult item. This continues until a score is reached that (a) has desirably low measurement error or (b) a specified time limit has been met. CAT has been used in educational assessments for over 40 years (Weiss, 1982). CAT uses deterministic algorithms and Item Response Theory rather than generative AI or machine learning. This approach provides transparent, interpretable adaptive testing without concerns about AI hallucinations or unexplainable decision-making.

In the present study and past work, validity claims and evidence are presented using a standards-based approach (AERA et al., 2014). Validity helps to answer how we know – what we know. A standards-based approach articulates claims justified by evidence tied to the five validity sources described in the *Standards for Educational and Psychological Testing* ([*Standards*], AERA et al., 2014). Those five sources of validity evidence are test content, internal structure, response process, relations to other variables, and consequences of testing/bias. Collectively, these five sources function like a braided rope: Greater evidence across numerous validity sources makes for stronger validity claims and in turn a robust validity argument for a test. This is akin to how rope braided with multiple rope fibers is stronger than a single rope fiber (King et al., 2025).

Previous studies with the paper-pencil PSMs (PSMs 3-8) offered claims related to all five sources of validity evidence. The transition from a paper-pencil format to a CAT delivered online required new investigations to support validity claims, which are summarized here. Bostic and colleagues (2024) shared three claims with justifiable evidence: The first claim was that the PSM-CAT items address mathematics content described in the CCSSM and are mathematically closed. The second claim is that PSM-CAT items adhere to the open, complex, and realistic framework. These two claims address test content. A third claim is that PSM-CAT items include limited bias across the developed items, which is tied to consequences of testing/bias. King and colleagues (2025) found a second consequence of testing claim: The benefits from using the PSM-CAT outweighed the negative outcomes from testing. Similarly, Koskey and coauthors (2023) state that bias was appropriately minimized across items, another claim. One group (e.g., male/female) did not have a strong bias in their favor that might impact their performance on the PSM-CAT. Related to internal structure: May and colleagues (2025) justify two claims: (1) The PSM-CAT measures middle-grade students' mathematical problem-solving performance with precision and reliability comparable to or exceeding that of the paper-pencil version. (2) The PSM-CAT consistently detects student growth in mathematical problem-solving performance across grades. The PSM-CAT functions psychometrically as well as its paper-pencil PSMs 3-8 counterpart, which meets modern standards. The final claim, associated with relations to other variables, was: There were no differences in performance by gender (May et al., 2025).

Claims and evidence for the PSM-CAT currently relate to four of five validity sources. The notable exception is response process. Response process is an idea that test takers produce a response that aligns with the construct and perform in ways that are expected (AERA et al., 2014; Bostic et al., 2021). Response process validity evidence is often gathered through think-alouds and cognitive interviews in both 1-1 and small-group formats (Bostic et al., 2021; Leighton, 2017). Recent syntheses highlight that response process evidence is sorely lacking in validation studies, with as little as 10% of mathematics education-related tests having any response process claims and evidence (Ing et al., 2024; Krupa et al., 2024). This proceeding’s research question fills a gap in the testing literature and rounds out the PSM-CAT validity argument: *What response process validity evidence exists for the PSM-CAT?*

Method

Edelen and colleagues (accepted) present validation as a methodology for STEM education. Validation brings together scholars, educators, and participants in ways to construct high quality assessments and tests that produce valid results and interpretations (Bostic, 2023; Edelen et al., accepted). There are no prescriptive methods for validation because the research purpose or question(s) drive the methods. Validation does not always require qualitative, quantitative, or mixed methods for every question yet any of these methods are available. The present study falls within the umbrella of validation, utilizing qualitative data and analysis for the research question.

Instrumentation: PSM-CAT & Think-aloud Items

The PSM-CAT comes from years of development with a design-science approach (Middleton et al., 2008). This development process included multiple rounds of reviews by mathematics educators, mathematicians, teachers and students, and bias panels consisting of students, K-20 educators, and community members. Items were developed to align with the Standards for Mathematics Content (CCSSI, 2010) and represent the mathematics domains. All items are written in English and meet grade-level reading standards as indicated by Flesch-Kincaid analysis (Bostic et al., 2024). A released expressions and equations-focused seventh-grade task is: "A water tower contains 16,880 gallons of water. Each day half of the water in the tank is used and not replaced. This process continues for multiple days. How many gallons of water are in the tower at the end of the fourth day?" The PSM-CAT is designed for students to complete as many items as they can within 30 minutes. Students take approximately 3-4 minutes per problem. Students are allowed to use scratch paper, writing utensils, and a calculator. The test contains a formula sheet and calculator though students may use a handheld calculator from their classroom. Any student receiving special services for a disability can complete the PSM-CAT so long as accommodation is followed. English Learners may complete the PSM-CAT so long as they do not receive special services for language learning (Cahill & Bostic, 2025).

For this study, students were interviewed using a 1-1 think-aloud approach (Leighton, 2017) and whole-class think aloud consisting of three students (see Bostic et al., 2021 for more information). Students were presented with a task one-at-a-time and asked to solve the problem. One task was printed on each sheet of paper. They could write on the task, had access to classroom calculators, and were encouraged to talk throughout the process. Students were reading directions, asked to agree to participate, then completed a practice task prior to the target tasks. Upon completion of the practice task, students were asked if they agreed to continue with the think-aloud. At the end of the interview, students were asked if they wanted to share anything with the researchers. Three problems were presented during each interview. Students took 20 minutes on average for a think-aloud. In total, 122 items were presented to grade six students, 128 items to grade seven students, and 111 items to grade eight students. Each item was seen by at least three students. Each item was seen by at least three students during three interviews.

Data Collection & Analysis

This study used representative purposeful sampling (Creswell, 2014). A goal with this sampling approach was to gather data from varying ethnicities; rural, suburban, and urban contexts; multilinguals and monolinguals; and students across various parts of the USA including the Midwest, Mountain West, and Pacific communities. Qualitative data for this study came from sixth-, seventh-, and eighth-grade students across these settings. Researchers worked collaboratively with teachers and school personnel to interview students representing different attributes (i.e., sex, ethnicity, and English-speaking status).

Qualitative data were analyzed using an inductive approach to draw out themes from the data (Miles et al., 2014). The inductive approach included multiple stages that was iterative at times. The data coding and analysis team included one university faculty and numerous students. Training was conducted prior to any analysis across team members and coders had high interrater agreement ($r_{wg} = 0.98$) that exceeded the threshold of 0.90 (James et al., 1993). Other research team members served as external others to triangulate the outcomes during each stage (Creamer, 2017). In brief, there were five stages of qualitative analysis. The first stage was reviewing students' work as well as rewatching video data to become familiar with the data. The second stage was recording general observations and making notes. These notes were transferred to a spreadsheet after each day of interviews. At the end of each interview day, coders co-reflected on the research question, tasks, student data, general observations, and notes. This led to the third stage: memoing based on these reflections. Memos were shared with other team members at the end of each interview day to corroborate the initial findings. Memos were retained until all items were administered and further analyses were conducted. The fourth stage required coders to take up these memos and analyze them further to construct potential themes and consider counterevidence. Potential themes were considered and reviewed in light of the data, honing them until a potential theme was clear and succinct. These potential themes were presented to other team members for triangulation. Theme communication was the fifth stage.

Findings

A central finding was that students responded to PSM-CAT items in ways that aligned to hypothesized patterns. There was a paucity of items that had any counterevidence to that theme:

one from grade six (0.8% of total items), three from grade seven (2% of total items), and none from grade eight (0% of total items). A transcription from one seventh-grade student named Carl serves as an example of the think-aloud data. Carl's data comes from his 1-1 think-aloud tied to the Water Tower task presented earlier.

Figure 1

Transcription of Carl's Work

Carl: A water tower contains 16,880 gallons. Each day half of the water in the tank is used and not replaced. This process continues for multiple days. How many gallons of water are in the tower at the end of four days? So, if it contains that much gallons of water then each day half of the water in the tank is used and not replaced. This process continues for multiple days. How many gallons of water are in the tower at the end of the fourth day? Wait so… [pause] that much… [pause] each day half of the water in the tank is used and not replaced. [starts writing on paper] First let's go 1…6…8…8…0. And then we go… divide that in half. So that's 8, and then 16. Zero, bring down the 8. That's 4…8…[bring down] another 8. That's 4… and bring down the zero. So, that's 8,440. Wait. [inaudible; erases some work] Okay, so then that would be in 1 day. So after 1 day, that's what it is. And then if I do it again: 8…4…4…0…divided by 2, that goes into 8 four times…[inaudible] bring down the 4. Goes in twice. Bring down another 4 Goes in twice. And then zero…goes in zero. Okay and then I would have to do that again because that's the third day. Wait yeah that's the second day. Do it one more time. 4…2…2…0. And then divide it in half. So, 2 goes into it [4] twice and that's 4. So that's zero. Bring down the 2. Goes into it twice, no - once! And it's two. So I bring down another 2. It goes into it once. That's another 2. Bring down the zero and that's zero. So, then it [answer] would be 2,110 gallons. [Pause]. Wait. [pause] No. I need to do it one more time because it says 'end of the fourth day' and I've got three days done. [writes down division similar to past work]. 1..0..5..5! 1,055 gallons at the end of the fourth day! [looks at problem] Yeah, that's it.

This transcript, along with hundreds of others, provided rich data. There was consistency across the sample, which included students from Midwest rural contexts, urban Pacific communities, as well as suburban contexts from the Midwest and Mountain West. Ineffective strategies and incorrect results also provided helpful responses in light of response process evidence. For example, incorrect results like 2,110 for the water tower problem demonstrate that a problem solver can perform the necessary procedures but struggle with interpreting the problem, which can lead to an incorrect result. For this proceeding, we provide an example of one student's think-aloud to imbue confidence that finding was reached in a valid manner.

Discussion

This research study provides a key claim and validity evidence related to the PSM-CAT and responds to calls for greater response process validity evidence. This PSM-CAT claim and evidence round out the previous claims such that all sources of validity evidence are addressed – something rarely seen in the mathematics testing space (Krupa et al., 2024). Educators and researchers may feel confident using the PSM-CAT.

Not only does this proceeding present evidence in relation to a response process validity claim, it provides a model for others to explore response process validity evidence. Gathering respondents' thinking prior to broad use is a means to solidify that test items function as intended

(AERA et al., 2014). A conclusion from this response process-focused study is a strong, metaphorical rope supporting the validity claims for the PSM-CAT.

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DEVELOPING GRADUATE INSTRUCTORS' COMPUTATIONAL THINKING IN UNDERGRADUATE MATHEMATICS

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This study explores how graduate teaching assistants (GTAs) developed computational thinking (CT) in undergraduate (UG) mathematics through a professional development model that integrates mathematical modeling (MM) tasks intended to support GTAs' CT practices such as decomposition, abstraction, algorithmic reasoning, and debugging. Data sources included pre-post-survey reflections. Findings suggest that engaging GTAs in MM deepened their conceptions of CT and supported their perceived readiness to integrate CT into UG math instruction. Implications include preparing future faculty to connect CT with mathematical reasoning and UG learning.

Introduction

This study examines the integration of computational thinking (CT) into undergraduate (UG) mathematics classrooms through the professional development (PD) of graduate teaching assistants (GTAs). Although CT has become increasingly prominent in K–12 mathematics education, its role in postsecondary contexts, particularly in developmental and introductory courses, remains underexplored. GTAs frequently serve as instructors in these courses, often entering with strong mathematical preparation but limited pedagogical training. To address this gap, we designed a PD model in which GTAs engaged in math modeling (MM) activities as learners, providing opportunities to engage with CT practices such as decomposition, abstraction, and debugging. With the assistance of AI and technology, this approach enabled us to examine how GTAs' engagement in MM activities that support CT relates to their perception of MM as a pedagogical resource for fostering undergrad students' CT through modeling-based instruction.

This study analyzes multiple data sources, including reflective journals and pre- and post-survey responses, through thematic coding informed by CT frameworks. The findings reflect GTAs' perceived readiness to use mathematical modeling as an instructional approach, grounded in their views of MM activities potential to foster CT among undergraduate students. The results contribute to preparing for the future faculty and positioning CT as a natural extension of mathematical thinking in UG classrooms and address the following research question: How does participation in a PD that integrates mathematical modeling shape GTAs' understanding of CT and their perceived readiness to integrate CT via MM into UG mathematics instruction?

Literature Review

Teacher Professional Development (PD)

Effective teacher PD is characterized by sustained engagement, content relevance, active learning, and collaboration (Desimone & Garet, 2015). Long-term, job-embedded PD allows teachers to reflect on their practice, integrate new knowledge, and apply strategies aligned with current standards and the needs of diverse classrooms (Darling-Hammond, 2017). Teacher beliefs about students, materials, and pedagogy develop through pre-service experiences and evolve via classroom practice and professional learning (Noben et al., 2021). Such beliefs can either foster or hinder instructional innovation (Cross, 2009). Sustained, reflective PD supported by collegial collaboration and school leadership can gradually transform these beliefs and promote reform-oriented practices (Cross, 2009).

Graduate Teaching Assistants (GTAs) play a vital role in higher education, supporting faculty, conducting lab sessions, and providing tutorial assistance. GTAs often enter teaching roles with limited pedagogical training, leading to variability in their instructional effectiveness (Gardner & Jones, 2011). Since the 1990s, many universities have devoted resources to GTA training programs, successfully focusing on generic teaching skills, discipline-specific strategies, and pedagogical skills (Chiu & Corrigan, 2019; Langdon, 2017). These have created better learning and teaching environments, increasing teaching competency, and improving the learning experience of UG students (Gardner & Jones, 2011). However, the typical duration of GTA training programs, ranging from a few days to a semester, constrains GTAs' development. Continuous support through mentoring, peer support, the commitment of senior academics to guide and support GTAs in teaching, and in-class observation help them address ongoing teaching challenges (Deshler et al., 2015). Previous research has reported that formal GTA training programs can boost self-efficacy in teaching and effective teaching behaviors (Chiu & Corrigan, 2019). However, extended studies on GTAs' self-efficacy beyond the initial pre-service teacher education program are limited, with mixed findings (Langdon, 2017).

Computational Thinking

Computational thinking (CT) has emerged as an essential component of mathematics education, emphasizing problem decomposition (breaking complex modeling tasks into manageable subproblems), abstraction (identifying essential variables while suppressing non-essential features), debugging (breaking complex modeling tasks into manageable subproblems), and algorithmic reasoning to enhance conceptual understanding. Examples of each include abstraction, which requires identifying essential variables (e.g., interest rate structures in the mortgage model) while suppressing non-essential features, debugging, when GTAs utilize AI to evaluate unexpected model behavior, and reviewing assumptions or formulas. Empirical research demonstrates that integrating CT within mathematics instruction promotes students' mathematical reasoning when computational tools are used to explore and represent ideas (Ye et al., 2023). Studies emphasize that CT is most effective when embedded in authentic mathematical contexts rather than taught as an isolated coding skill. Additional research demonstrates that CT can deepen students' representational fluency and support model-based reasoning in mathematics and STEM (Berland & Wilensky, 2015; Weintrop et al., 2016).

Novice teacher preparation remains pivotal for meaningful CT integration. Dong et al. (2024) found that project-based and course-embedded CT modules enhance teachers' abstraction, debugging, and problem-solving abilities while increasing self-efficacy. Similarly, Rodrigues et al. (2024) identified that many teacher education programs offer brief or theory-heavy CT preparation without sufficient classroom application, highlighting the need for extended, practice-oriented experiences. Butler and Leahy (2021) demonstrated that constructionist approaches in digital learning modules improve pre-service teachers' understanding of CT, though more extended engagement yields deeper pedagogical integration.

Overall, research converges on the need for sustained, practice-based PD that merges computational and mathematical practices. This aligns with work demonstrating that CT instruction is most effective when embedded in authentic disciplinary contexts rather than isolated activities (Weintrop et al., 2016). GTAs can also benefit from such frameworks through scaffolded co-teaching, feedback, and exposure to CT-rich instructional models.

Despite progress, gaps persist in longitudinal evidence and validated assessment tools for CT in mathematics. Additionally, the intersection of CT and AI in educational settings remains underexplored, despite AI's growing influence on how students engage with computational problems in mathematics. This gap is particularly relevant as our findings reveal AI-mediated learning contexts where traditional CT frameworks may require adaptation. Continued research is needed to refine teacher and GTA training models that cultivate computationally enriched mathematics instruction grounded in classroom practice.

Methodology

Participants were GTAs enrolled in a public university mathematics department in the southeastern United States. A multi-stage PD structure was designed to align with GTAs' evolving needs and to avoid overburdening them by separating exploration, design, and implementation across stages. This staged design enabled close examination of GTAs' experiences over time and informed iterative refinement of the PD. Because the purpose of this study was to explore shifts in GTAs' understanding of CT and their perceived readiness to integrate CT through MM into undergraduate mathematics instruction, a case study approach was selected to capture the uniqueness and complexity of GTAs' engagement with MM and CT.

The case study followed four GTAs who participated in a PD embedded in their teaching assistantship; one withdrew mid-semester, thereby analyses are based on the remaining three participants. GTAs engaged in a 15-week co-teaching model with faculty that included weekly workshops on lesson planning, assessment, and instructional technology to support instructional confidence and reflective practice (Noben et al., 2021). The first stage consisted of a five-week special topics PD on CT, using MM as a context for exploring instructional integration; GTAs were encouraged to use AI as a supportive resource where appropriate. In future subsequent stages, GTAs will teach their own courses while participating in PD modules on lesson design and enaction emphasizing inclusion of computational tools, e.g., Desmos and spreadsheets, asynchronous learning, guided inquiry, student-centered instruction etc.

Data Collection

We collected data through several stages of the PD program. Before the PD sessions began, a baseline for the GTAs' understanding of CT was established through a pre-survey comprising 19 Likert scale items assessing familiarity with CT terminology, confidence in using CT skills along with two open-ended questions designed to capture their initial perceptions and uncertainties about integrating CT through mathematical modeling. The post-survey expanded the open-ended component to capture shifts in GTAs' perceptions following participation in the PD. Newly added questions prompted GTAs to reflect on the perceived value of mathematical modeling for addressing real-life problems, anticipated benefits and challenges of using modeling tasks to support students' learning and computational thinking, and the specific computational thinking practices they engaged in or observed as salient for students. Additional items elicited reflections on the role of AI in modeling activities, including GTAs' experiences using AI during a modeling task and their perspectives on allowing students to use AI as part of mathematical modeling instruction. We analyzed observational data collected during sessions, and the post-PD open-ended questionnaire was analyzed to identify convergent and divergent themes.

Data Analysis

Data were analyzed using thematic coding informed by Braun and Clarke's (2006) framework to identify patterns across participant responses to a structured reflection protocol. The dataset consisted of qualitative responses from three GTAs, who participated in a professional development (PD) program focused on integrating CT into undergraduate mathematics instruction through modeling tasks. Each GTA responded to 12 open-ended post-survey questions related to their experiences with the mortgage modeling task, understanding of CT and mathematical modeling, and integration of AI tools. Through inductive coding and cross-case analysis, patterns across the participants' reflection were identified to examine GTAs' experiences and perceptions related to CT, mathematical modeling and AI.

Preliminary Findings

Preliminary analysis draws on both pre- and post-survey GTA reflections to examine shifts in GTAs' perceptions of CT and MM, as well as their perceived readiness to integrate CT through MM into UG math instruction following participation in the PD. Pre-survey responses revealed substantial uncertainty regarding both the nature of CT and how MM could be enacted in classroom contexts. One GTA noted being "not quite sure exactly how computational thinking differs from 'regular' thinking," while another expressed uncertainty about "what math modeling would look like" in their class. Although a third GTA indicated attempting to include some modeling content, they questioned how such activities would translate into students' holistic understanding of CT, reflecting early interest alongside limited instructional clarity and confidence. The following section presents post-survey findings that further illustrate how GTAs' perceptions of CT and MM evolved through participation in the PD.

Computational Thinking: Pedagogical Resource Amid Cognitive Complexity

All three GTAs recognized CT as a foundational skill for mathematical problem solving and instruction. CT practices, particularly decomposition, pattern recognition, abstraction, algorithmic reasoning, and debugging, were consistently identified as integral to the modeling tasks. For example, one GTA noted, “In order to model, you have to use computational thinking skills. Particularly, you need decomposition (making assumptions), often pattern recognition, 100% need abstraction to make an efficient model, you often need algorithm design, and definitely use generalization.” Two GTAs noted that CT was both cognitively necessary but also pedagogically valuable, with modeling tasks serving as authentic contexts for CT development. All three GTAs anticipated that students would struggle most with abstraction and debugging, citing cognitive overload and lack of routine structure as barriers.

Mathematical Modeling as a Vehicle for CT

All three GTAs unanimously affirmed the value of mathematical modeling in fostering relational understanding, perseverance, and real-world relevance. Modeling tasks were seen as effective tools for helping students break down complex problems and apply mathematics meaningfully. One GTA mentioned, “It helps develop a better understanding rather than just plugging values into a formula.” Despite these benefits, time constraints and pacing challenges emerged as significant instructional hurdles. All three GTAs expressed concern about balancing modeling depth with curricular coverage, especially in introductory courses.

AI Integration: Support and Risk

All three GTAs primarily used AI for verification and exploration of alternative solution paths. Two GTAs found AI explanations helpful; the other GTA reported that responses were overly technical or misaligned with their needs. Two GTAs cautioned that premature AI use could undermine productive struggle and recommended integrating AI only after students first engaged independently, ensuring their thinking was not replaced by automated assistance. One GTA highlighted this concern, stating, “I worry that using AI might rob them of using/developing CT skills. They might not push past struggling to get to a point of understanding, and instead just put the part they're stuck on in AI.”

Other Emergent Themes

Several additional themes emerged. Emotional responses to the modeling tasks ranged from frustration to accomplishment, often influenced by peer pacing and task clarity. All three GTAs reflected on their own instructional practices, noting a shift from modeling CT themselves to considering how to engage students more directly in CT processes. The PD model employed prompted meaningful reflection and engagement, supporting the development of CT as both a cognitive and pedagogical resource. Emergent themes indicate that the PD supported conceptual growth in CT and modeling, which is a necessary precursor to designing CT-rich instruction. Although this stage focuses on epistemological development and reflective stance, future analysis will address how these understandings manifested in GTAs' instructional artifacts.

Conclusions, Limitations, and Next Steps

This study examined how GTAs' engagement in PD experience centered on mathematical modeling informed their understanding of CT and perceptions of readiness to integrate CT within UG mathematics instruction. Participants demonstrated a growing awareness of CT practices and recognized the value of mathematical modeling as a vehicle for fostering these skills in students. While AI was seen as a helpful tool for verification and exploration, concerns similar to those highlighted by Abubakar et al. (2025) about overreliance and diminished student engagement in problem-solving were raised. Overall, the study highlights the need for intentional support structures, including opportunities for reflection and adequate time to engage with modeling tasks, to help GTAs navigate the conceptual and pedagogical complexities of modeling, CT, and emerging technologies in mathematics education.

Several limitations should be considered. The small sample size ($n = 3$) and single-institution context limit generalizability. The preliminary analysis relies primarily on self-reported reflections rather than observed classroom practice, potentially overestimating actual instructional changes. One participant's mid-research withdrawal may have introduced selection bias, conflicting viewpoints, or other data points that could have altered overall conclusions.

This preliminary report only considers the GTA reflections of the PD process; future analysis of enacted practice (e.g., student work and GTA lesson plans) will be derived from extended classroom observation. Additionally, future research should consider including larger samples along multi-institutional contexts. This would assist in strengthening potential generalizability.

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FOSTERING PRODUCTIVE STRUGGLE IN STATISTICAL LEARNING THROUGH DYNAMICAL STATISTICAL SOFTWARE

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This study investigates how dynamic statistical software fosters productive struggle in learning statistical variability, particularly for prospective teachers. Video analysis of classroom interactions reveals how prospective teachers progressed from viewing variability as mere formula memorization toward deeper conceptual understanding. Findings demonstrate that technology-supported productive struggle maintains cognitive demand while enabling prospective teachers to construct meaning independently, offering valuable insights for statistics education that moves beyond procedural instruction.

As society becomes increasingly shaped by data, the need for students to develop robust statistical thinking becomes pressing. A core—and often challenging—component of this thinking is variability: understanding how and why data points differ, and what that variation means. However, variability is often reduced to memorizing formulas rather than taught conceptually, due to limited instructional focus and gaps in teacher preparation (Reading & Shaughnessy, 2004). As a result, many mathematics teachers are asked to teach statistics courses without adequate background knowledge (Reading & Shaughnessy, 2004), limiting their ability to foster students' conceptual understanding and support productive struggle. The *Guidelines for Assessment and Instruction in Statistics Education* (GAISE I&II; American Statistical Association, 2005, 2020) emphasize the use of technology to support meaningful engagement with data. With technology, we can develop tailored lessons to each learner's specific needs. Dynamic software environments, such as *TinkerPlots* or *CODAP*, provide a space where students can explore data structures visually and conceptually. Unlike software used by professionals (e.g., R, Python), *TinkerPlots* or *CODAP* was intentionally designed to support elementary and middle students, offering drag-and-drop functionality, intuitive graphing tools, and visual indicators that promote deeper reasoning. This study investigates how students interact with such technology to confront and overcome intellectual challenges in learning about variability, focusing on how productive struggle unfolds in this dynamic digital context.

Literature Review

Productive struggle, as defined by Hiebert and Grouws (2007), refers to the intellectual effort students exert when making sense of complex concepts that fall within their zone of proximal development (Vygotsky, 1978). Rather than avoiding difficulty, students are encouraged to persist through confusion and explore alternative strategies. This is essential as students make meaning through productive struggle, or as they grapple with mathematical ideas that are within reach, but not yet well formed (Hiebert & Grouws, 2007; Warshauer, 2015). Warshauer (2015) claimed that while struggles are situated with the task and student, struggles can be directed more productively by teachers' support. She identified a struggle to be productive if the cognitive demand of the task is maintained, student thinking is supported, and student actions were enabled to move forward by teachers' responses and questions. However, other research has found that teachers tend to provide struggling students with procedures or lower-cognitive demand support (Santagata, 2005). As claimed, the students implement the method suggested by the teachers, and there is no need for them to engage in any further exploration, thus diminishing the struggle and losing the opportunity for "intellectual challenge". And this "intellectual challenge" is the key to developing students' agency and autonomy in learning mathematics (Yackel & Cobb, 1996).

Technology invites a revisit of what productive struggle should look like and how to support such struggle in an alternate way. Technology might reduce surface-level barriers, such as visualizing some concepts or initial ideas, freeing up students to spend more time exploring or resolving the task. With timely feedback and guidance through technology, the actions students try take to overcome their difficulties ("agency") will be more "autonomous" and "productive". The working definition of mathematical agency used in this paper is the capacity to take purposeful action when confronted with mathematical challenges. Mathematical autonomy is a sense of ownership over one's mathematical reasoning, which can manifest as the willingness to question or resist the reasoning of mathematical authorities (i.e., instructors, carefully designed technology, textbooks, or expert peers). Carefully designed technology tools like *TinkerPlots* or CODAP can be viewed as mathematical authorities (Ben-Zvi & Arcavi, 2001; Cobb & McClain, 2004), as they offer ways for students to test hypotheses, explore data visually, and refine their conceptualizations without immediately relying on teacher intervention.

Previous research on student struggles has been limited and has primarily focused on examining whether struggle occurred without examining in detail what helps students to overcome their struggles. Although the idea of the importance of struggle in learning has been examined in solving algebra problems (e.g., Warshauer, 2015) in the middle grades, there has been sparse research in settings other than middle-grade mathematics classrooms and how instructors could support those productive struggles without lowering the cognitive demand of the tasks, especially with the help of technology. This study is guided by the following research questions: How do students leverage the dynamic features of these technological tools to navigate and overcome impasses and engage in more productive struggle?

Methods

This study was conducted in a statistics content course for pre-service teachers at a large West Coast university. Of the 28 enrolled students, 23 consented to participate. Data collection focused on small groups (2-3 students) where all members were participants, with each group provided a laptop and access to an interactive whiteboard. Two primary data sources were analyzed: (1) video recordings and observational field notes of group work, and (2) student-generated artifacts (assignments, projects). The core task analyzed was the "Cats Data Exploration," a statistical investigation aligned with GAISE recommendations (American Statistical Association, 2005, 2020) that emphasizes conceptual understanding through active, technology-supported inquiry. Using a real dataset of 24 cats, students posed their own questions (e.g., "Do male cats have longer body lengths?") and used *TinkerPlots* to explore distributions and variability. They employed drag-and-drop features to create and manipulate plots (e.g., dot plots), calculate measures of center, and visually compare groups, all while drawing on preliminary concepts of range and spread.

A qualitative, exploratory case study design was employed to examine *how* students navigated impasses and engaged in productive struggle with technological support. From approximately 10 hours of video, moments of "struggle" were identified based on verbal (e.g., "I'm stuck," "I don't know how to…") and non-verbal (e.g., prolonged silence, confused gestures) indicators. For small group discussion in particular, students raised hands or asked peers when they got stuck or were unsure how to proceed. An episode began when a student or group encountered an obstacle and ended when they resolved it or moved forward meaningfully. A total of 27 distinct episodes were identified across the recorded sessions.

An initial coding scheme was developed deductively from the literature on productive struggle (Warshauer, 2015) and the *Responsible Use of Educational Tools Framework* (Griffin & James, 2025) and then refined inductively through multiple viewings. The final scheme included the following primary codes:

- Nature of Struggle: *Technical* (tool-related), *Conceptual* (variability-related), *Representational* (graph choice/interpretation).

- Strategy Used: *Peer dialogue*, *Tool experimentation*, *Instructor query*, *Revisiting prediction*.
- Agency/Autonomy Indicator: *Self-initiated action*, *Questioning the tool/output*, *Justifying a choice*, *Ownership statement*.
- Resolution Type: *Tool-mediated insight*, *Peer-supported breakthrough*, *Continued uncertainty*.

The author and two trained graduate research assistants independently coded all 27 episodes using the finalized scheme. Discrepancies were discussed in weekly calibration meetings, leading to clarifications in code definitions and the merging of two initially separate categories. The coding scheme was thus refined iteratively over three cycles until consensus was reached on all episodes.

Results

The following section presented two episodes in which students overcame impasses to demonstrate the importance of dynamical statistical software to help students engage in more productive struggle when learning variability concepts. In this statistical investigation, they were trying to compare the body length of female cats and male cats.

Episode 1

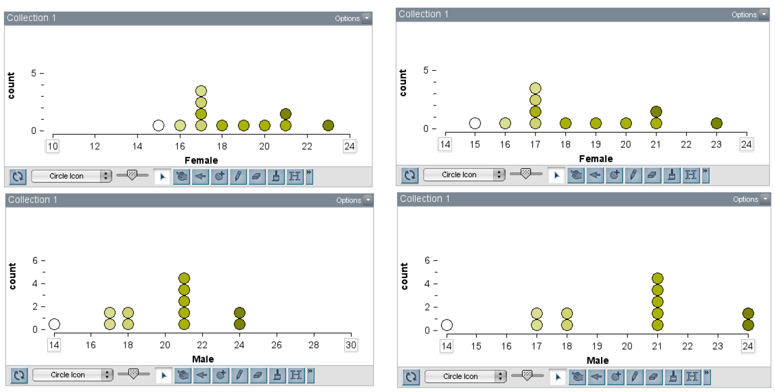
When I was passing student C and D, I saw they had two displays below, one for female cate and one for male cat. They raised their hands for help.

> Student D: Our graph show us female cats will in general have longer body-length. Is that right?
>
> Student C: That's different from what other groups got…I saw they got something different…but that's what our graph show us…we are stuck…
>
> Teacher: If you read the min and max value from each of the graph, what are they?
>
> Student D: 15 is the min for female and 23 is the max for female. And 14 is the min for male and 24 is the max for male. Wait! Male cat has a max body length larger than female? We must have done something wrong…
>
> Student C: I see it! The location of 24 is different on the two graphs! Did you see that? Here (point to the 24 on the female data graph in Figure 1) is the end but here (point to the 24 on the male data graph in Figure 1) is more than half of the x-axis. We need to change our axis so the 24 is locate at the same place.

We can see those two students engaged in productive struggle when *TinkerPlots* helped them to construct their two graphs on the same scales. In the beginning, both of them assumed that female cats have longer bodies than male cats because of the mismatched scales in both graphs. After finding the maximum and minimum values for both graphs, they realized that the locations of 24 differed between the two graphs, so they needed to change their x-axis. With the axis tool in *TinkerPlots*, they successfully changed the two graphs to the same scale, and that helped them to visualize that there are more male cats with body lengths longer than 21 compared to the female cats. *TinkerPlots* visually highlights the importance of having the same scale on graphical displays easily when describing the variability between two data sets.

Figure 1

Student C And D's Initial Comparison Plots (Left) and Adjusted Plots (Right)



Student D: Right…right…Professor, how do we change the axis?

Teacher: If you double click the box in the beginning and the end and change them to the same number…

Student C: Change them to both starting at 14 and endings at 24…Now it looks different from what we have before. There are more male cats has body length more than 21 than female cats!

Episode 2

Students E and F successfully constructed the dot plot. But I saw they were clicking every data point to try to remember which one was the female cat and which one was the male cat. They raised their hands for help.

Student F: We are trying to figure out whether female cats have longer body length or male cats has longer body length…but we need to click on each data point to find out and didn't go far…is there an easier way of doing that?

Student E: Our plot seems to only show the body length so we need to click each data point to see whether the point is female or male…how can we put the gender variable also into the graph?

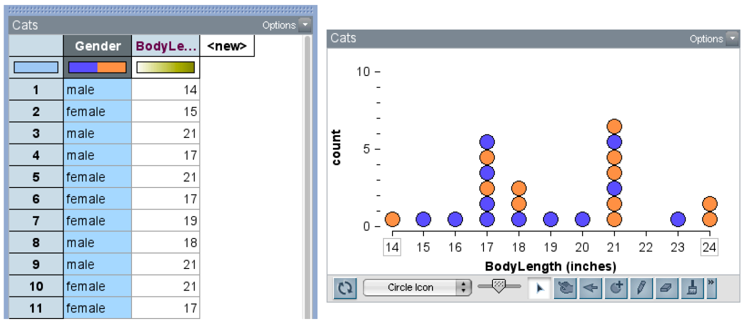
Teacher: What if you click on the gender variable and see what it shows you?

Student E: Click on the gender variable…you mean each of the point?

Student F: Or maybe the word "gender" (pointing to the word gender in the table of Figure 2) …what if we click that…. WOW! The color shows here! That's a magic.

Figure 2

Student E and F's Two-Variable Plot of Female and Male Cats' Body Length by Gender



Student E: Do you know which one is which (looking at F)? Is the blue male or female (pointing to the graph in Figure 2)?

Student F: That's a good question… I have no idea…Professor, is there a way to find out which one is which?

Teacher: There are multiple ways, what if you try and drag any of the point a bit up and try to separate the color?

Student F: Let me try…drag any point, I will drag this orange point (the first orange point on 18 in Figure 2) up…they separated into two groups!

Student E: WOW! Magic again…I like this…it actually shows us the male is orange on the top and the female is blue on the bottom…cool…now it's so easy for us to make the inference.

Teacher: So what is the inference that you can make based on this graph?

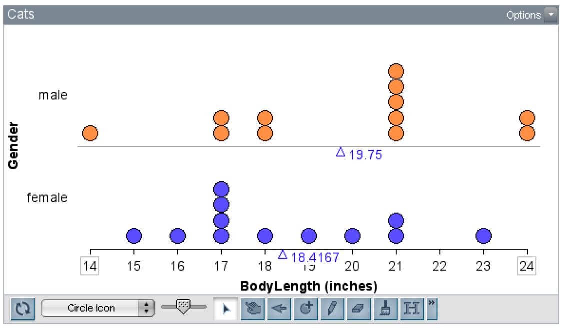
Student F: Male cats seem to have longer body length since they have more point on the right side of the graph…female cats are more "clustered" on the left side of the graph, like around 17 (pointing to the graph in *Figure 3*).

Student E: I remember we can also look at their mean and their average distance to the mean…where is it (looking at the tool buttons on top of *TinkerPlots*) …I can do this…I see! If I click this "mean" button, it shows me the mean for the male cats is 19.75 and the mean for the female cats is 18.1467…That means that the majority of the data for female would be not far away from 18 and the majority of the data for males would be not far away from 19. So the majority of the male cats will have longer body length than the female cats. Oh my god! We are such geniuses.

We observed that these two students engaged in productive struggle once they linked the second variable into their graphs. There is no doubt that E and F would be able to make a prediction based on what they have in the first graph they made, by clicking each of the data points to figure out which one is the female cat, and which one is the male cat. However, that required a lot of memorizations, and if they explain their reasoning to other students in class, others might not buy what they say based on their first graphs. Just by clicking the second variable itself, *TinkerPlots* was able to show the variable "gender" in two colors. Even more, after using the drag tool in *TinkerPlots*, students successfully separated the female cats and male cats into two groups, in this way, the two groups will be displayed on the same scale of axis. Based on that, E and F successfully made their prediction that male cats tend to be longer than female cats based on the distribution of the data, measures of center, and the spread of the data.

Figure 3

Student E and F's Adjusted Plot of Female and Male Cats' Body Length by Gender



Conclusion and Implications

This study explored how technology-mediated statistical investigation helps prospective teachers shift from viewing variability as a procedural formula to understanding it as a dynamic concept. It specifically examined how students overcome conceptual struggles with minimal direct instruction—a gap in existing research (Santagata, 2005). The instructor used carefully framed questions instead of procedural guidance to maintain cognitive demand, aligning with Warshauer's (2015) framework for productive struggle. The three required conditions—preserved demand, supported thinking, and forward progress—were each observed as students engaged in peer dialogue, tested ideas with *TinkerPlots*, and refined their reasoning about variability. Episodes from the 'Cats Data Exploration' illustrate how this worked in practice. When students struggled to create comparable graphs, the instructor's subtle cues about *dragging* and *stacking* helped them construct appropriate displays without lowering demand. When they were uncertain how to compare groups, the *axis* and *attribute separation* tools allowed them to visually align datasets and differentiate distributions. In each case, *TinkerPlots* provided an authoritative yet non-directive source of feedback.

The findings highlight the broader educational value of dynamic visualization tools for building statistical literacy and visual reasoning across K–16 STEM curricula. By providing accessible, inquiry-based pathways to complex concepts, such tools can promote equitable learning opportunities, especially for students who may struggle with purely symbolic or procedural approaches. Integrating these strategies into teacher education is therefore essential for preparing educators to teach statistics in ways that prioritize conceptual understanding.

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MODIFIED READING COMPREHENSION STRATEGIES AND MATHEMATICS WORD PROBLEM SOLVING

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Solving mathematical word problems requires two kinds of understanding: reading and mathematics. This theoretical paper hypothesizes that word problems structured into a 3-sentence template with the modification and integration of two reading comprehension strategies, Question Answer Relationships [QAR] and FAB4, during step 1, (understand the problem), of Polya's problem solving process may promote success. First the grounding of this work is discussed, followed by the description of the proposed model, and operationalizing problem comprehension with mathematical diagrams. Future research methodology focuses on assessing and evaluating this integrated conceptual model.

Many researchers explore mathematics teaching practices that draw on literacy with journaling (Kostos & Shin, 2010), class discussion (Smith & Stein, 2018) or comprehension in mathematics from a reading perspective (Fuentes, 1998) to bolster mathematics performance. Reading comprehension "involves the construction of a coherent mental representation of the text in the readers' memory" (van den Borek et al. 2014, p. 10). Mathematics comprehension requires cognitive processes (Jaffe & Bolger, 2023) for meaning constructing or *mathematizing,* in which students "take their initial focus off the specific numbers and computation and put their focus on the actions and the relationships expressed in the problem" (Moore et al., 2020, p. 3). The reader shifts from reading comprehension to *problem comprehension*, which requires the reader to translate "each sentence into an internal representation and integration of the information into a coherent structure" (Lewis & Mayer, 1987, p. 363).

I contend in step 1 of Polya's (1973) 4 step mathematics problem solving process, Understand the Problem, Devise a Plan, Carry Out the Plan, and Look Back, should include a process that involves a 3-sentence template, 2 modified and integrated reading comprehension strategies, and student generated mathematics diagram. Former work focused on modification of the QAR strategy and the 3-sentence structure with the FAB4 integrated into QAR. The new work will test the integration of the two strategies with the 3-sentence structure and will have students create mathematics diagrams to ascertain their problem comprehension competency.

Literature Review and Grounding for the Conceptual Model

In this section, I describe the 3-sentence template, the original reading comprehension strategies, and their modifications.

Three Sentence Template

Informed by Lewis and Mayer (1987), I offer a three-sentence template that supports early readers' shift from reading comprehension to problem comprehension. Sentence 1 introduces the first object using subject-object sequence. Sentence 2 introduces the second object and its relationship to object one, using subject-object sequence. Sentence 3 asks the question (Mistele, 2023).

Reading Comprehension Strategies

I discuss two groups of reading comprehension strategies: Question Answer Relationships (QAR) and Fab Four (FAB4). They are first described for reading text followed by their modification for mathematics word problems.

QAR Strategies

QAR strategies help teachers elevate all students' literacy skills, which includes fiction and non-fiction when "identifying key details in texts, graphs, photos, and other materials (Raphael & Au, 2005, p. 207). The frame for QAR is developmentally based and begins with the location of information needed to answer the question, "In My Head" or "In the Book" (p. 210). Each of these types has two sub-levels describing the question-and-answer relationship. In My Head has two options, Author and Me and On My Own. In the Book has, Right There, and Think and Search.

In My Head – Two Types. Raphael and Au (2005) explain that the first type, On My Own questions "has the information in the text, but the answer is not" (p. 212). The student reads the text to answer the question by using their own ideas and experiences. Author and Me questions require the students to think about how the text and their prior knowledge fit together. This means the students are working with the author to answer the question.

In the Book – Two Types. The first type, Think and Search, means the answer is in the text. Readers think and search to find the answer or join distinct parts of the text to find the answer. The answer is found within a paragraph, across paragraphs, or across chapters in a book. There are literal questions, and the answer is in one location. Words from the question and words that answer the question are often "right there" in the same sentence (Raphael & Au, 2005, p. 212).

Modifying the QAR Framework for Mathematics Word Problems

In My Head – Two Types. Author and Me become *Problem and Me*. Similarly, the answer joins together information in the problem with the student's prior knowledge. Students combine and perform calculations from multiple lines in a table or interpret information requiring multiple steps to solve the problem (Mistele& Hilden, 2021). It may require complex thinking, and/or going beyond basic reading. On My Own questions are uncontextualized problems. Students use memory to perform algorithmic operations or procedures with automaticity (Mistele & Hilden, 2021). For example, the student knows how to add ¼ + ½ , or they do not. There is no context to help the student solve the problem.

In the Book – Two Types. Think and Search questions for mathematics word problems are the same for reading because the student either reads across information in a problem or in a graphic, but it does not require additional operations. For example, students read across multiple lines of a table, and/or complete a pattern (Mistele & Hilden, 2021). The modified Right There questions contain the answer to the question asked. It is linked to a graphic or diagram like a statistical graph (Mistele & Hilden, 2021). A small pilot experimental study using these two types found no statistical difference between the control group and the intervention group (Mistele & Hilden, 2021).

FAB4 ***Reading Strategies***

The FAB4 are research-based discussion techniques that are a Response to Intervention (RTI) plan for struggling students that focuses on metacognition (Oczkus, 2010). The strategies include *Predicting, Questioning, Clarifying*, and *Summarizing.*

Predicting. This is a type of guessing that includes previewing the text, such as looking at illustrations on a page or the books' cover and reading the title. The students use the frame: "I think this is about…because…" (Oczkus, 2010, p. 18).

Questioning. This strategy uses a variety of questions: wondering questions, quiz questions, inference or main points questions, and self-created discussion questions. (Oczkus, 2010).

Clarifying. Students monitor their understanding and use strategies to fix comprehension lapses that require they admit they fail to understand something and attempt to figure out how to remedy the lapse. This goes beyond lack of word recognition. Using the frame: "I didn't get the sentence … so I …" (Oczkus, 2010, p. 21) helps students locate where their comprehension lapses. Other prompts include: "I didn't understand the part where…", "This doesn't make sense…", "I can't figure out…" (Oczkus, 2010, p. 21).

Summarizing. Oczkus, (2010) claims this improves overall comprehension who cites Duke and Pearson, (2002). Mini summaries during the reading process help students recall notable events, details, order points, or synonyms used to enhance comprehension.

Modifying the FAB4 ***for Mathematics Word Problems***

Modifying the FAB4 (Mistele, 2023) for mathematics word problems may promote metacognition and sense-making through discussion that currently exist in many mathematics classrooms (e.g., Smith & Stein, 2018).

Predicting. For example, content that has graphs, charts, or diagrams in the word problem, students use the frame, "I think this is about…because…" (Oczkus, 2010, p. 18) supports comprehension because the student can point to the graphic. This can help students identify the QAR problem type while probing the problem.

Questioning. Students identify what they fail to comprehend by asking questions about the context of the problem, the relationship between the objects, or the characters in the word problem. For example, "How do the number of marbles Tom has, relate to the number of marbles Joe has?" Student questions may help with their comprehension gaps.

Clarifying. Many of the frames used for reading can be used with word problems. For example, "I reread the parts I don't understand, I think about what I know, or talk to a friend" (Oczkus, 2010, p. 22). These frames, currently used, help students to identify specific parts of the problem that are troublesome.

Summarizing. Summarizing and mini summaries during the problem-solving process help students verbalize the context of the problem, identify the action occurring in the problem, and/or the relationships between the objects.

The Enhanced Conceptual Model

Next, I describe the enhanced conceptual model that integrates the FAB4 into QAR with the 3-sentence template and introduces mathematical diagrams that intend to operationalize problem comprehension.

Integrating FAB4 into QAR

Integrating the FAB4 into QAR begins with identifying the QAR type for the mathematics word problem followed by the FAB4, which are flexible to address students' prior knowledge, reading and/or mathematics skills separately or simultaneously as shown below.

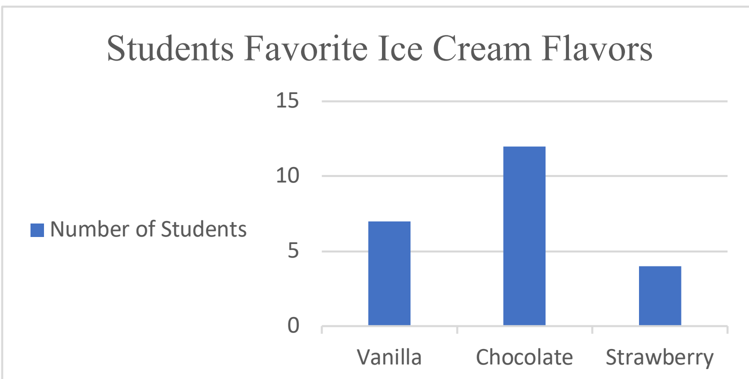
Right There Example

This is a Statistics and Probability content strand word problem, using the 3-sentence template.

> Ms. Martinez' class created a graph showing the student's favorite flavor of ice cream. (Initial condition sentence)
>
> Their favorite ice cream flavors include vanilla, chocolate, and strawberry. (Relational sentence)
>
> What flavor do most students in Ms. Martinez' class like? (Question sentence)

Figure 1

Graph Shows Data for Ice Cream Preferences



Students use the FAB4 as follows. *Predicting:* "I think this problem is about statistics because I see a graph." Students may identify this as a Right There problem, or a Think and Search problem. *Questioning:* "What do the numbers on the left side of the graph mean?" or "How are the numbers on the left side of the graph related to the columns?" *Clarifying:* students may re-read the problem again or talk to their friends to help them fill their comprehension lapse. *Summarizing:* students discuss what happened in the context of the problem or how the data relates to the graph. They discuss the different heights of the columns and what that means and/or the categories identified at the bottom of the graph. They identify this is a Right There problem and they look for the highest bar then move their eyes down to the category label to find the answer is chocolate.

Think and Search Example

Using the problem above, a new question can place it into a different QAR type as shown below.

> Ms. Martinez' class created a graph showing the student's favorite flavor of ice cream. Their favorite ice cream flavors include vanilla, chocolate, and strawberry. How many more students like chocolate ice cream than vanilla ice cream?

The student uses the FAB4 frames as they did above for *Predicting*, *Questioning*, *Clarifying* and *Summarizing*. Students probe the problem with the FAB4 and identify this as a Think and Search problem. They locate the vanilla bar that stops at 7. Next, they locate the chocolate bar that stops at 12. Most times, students use the *add on* technique to find the solution, counting from 7 to 12 to find the answer 5. Other students may use subtraction, 12 – 7 = 5.

Problem and Me Example

These problems require students to use their prior knowledge with the information in the problem to find the solution. The following is a Number and Quantity example.

> Chantel wants to bake some cookies for the Rescue Mission fund raising event.
> She has 6 cups of sugar in her cupboard. (Initial condition sentence)
> The recipe calls for 2 cups of sugar for one batch. (Relational sentence)
> How many batches can Chantel make? (Question sentence)

Students use *predicting:* "I think this is a multiplication problem because the question asks, how many batches." They may recognize this as a Problem and Me problem type. *Questioning:* "How are the numbers for the 6 cups that is in the cupboard and the 2 cups for one batch related?" or "I don't understand this part with the 2 cups and the 6 cups." *Clarifying*: the students may re-read the problem several times and talk to a friend. *Summarizing:* students briefly describe the action in the problem or use tools to help them act it out. Encouraging students to use mathematical diagrams or tools supports sense making. In the end, students recognize this as a Problem and Me. They must do something with the information in the text and use their prior knowledge about multiplication to answer the question.

On My Own Example

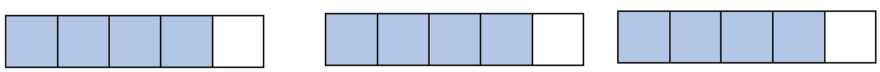
This is a non-contextual problem, $3 \times \frac{4}{5} = ?$ and students recognize this as an On My Own problem. The FAB4 helps them solve problems that rely on their prior knowledge. *Predicting:* "I think this is about multiplication because I see the multiplication sign." *Questioning:* "I wonder what this symbol means," for example. *Clarifying:* Students talk to a friend to help them recall the multiplication operation with a diagram or use the procedure. *Summarizing:* Students may recall the “groups of” model of multiplication that describes the relationship between the numbers, which may lead to a mathematical diagram shown in figure 2.

Introducing Mathematical Diagrams

Mathematical diagrams will be used to assess problem comprehension. I will use De Toffoli’s (2022) definition, “systemic notional item used in mathematics that is either geomantic/topological or two dimensional or both (p. 12) and her three criteria for a mathematics diagram includes, 1) is it possible for someone with appropriate training and tools to reliably reproduce a diagram, 2) the constitutive perceptual features are easily identifiable and carry mathematics content reliably, and 3) their constitutive uses correspond to well-defined mathematical operations (p. 22).

Figure 2

Groups of Model for Multiplication



The problems shared above show the flexibility of integrating the FAB4 within QAR for various mathematics strands and grade levels to support reading comprehension and problem comprehension.

Proposed Methodology for Future Research

An experiment methodology with a control group will test this concept with third grade students using In My Head problems. Mathematics diagrams will operationalize Lewis and Mayer’s (1987), problem comprehension. A team of experts will validate and then evaluate students’ mathematics diagrams using a new assessment instrument to evaluate the mathematics diagrams based on appropriateness, completeness, and accuracy with ranking from 0 – 2 in each category.

Conclusion

The modified QAR identifies the type of question asked in the mathematics word problem, and the modified and embedded FAB4 supports reading and problem comprehension simultaneously when using the 3-sentence template. Mathematics diagrams may operationalize problem comprehension that may prepare students to enter step 2, *Devise a Plan* (Polya, 1973). Further research will use an experiment methodology that integrates the modified reading strategies and operationalizes problem comprehension with a new tool to evaluate students' understanding.

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ACADEMIC LITERACY WITHIN FOUR PROBLEM-SOLVING PROGRESS SCENARIOS INVOLVING FRACTIONS

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This study examines how third-grade students use academic literacy to make sense of fractions during one-on-one teacher interactions in an equal sharing lesson. Drawing on classroom observations and interdisciplinary analysis, we identify rich learning opportunities across four problem-solving scenarios.

Introduction

Opportunities to learn mathematics with deep understanding are socially situated and participatory (Greeno, 2011; Moschkovich, 2002). Guided by this perspective, we ask: *How does academic literacy, enacted through children taking up space, appear across four fraction problem-solving fraction scenarios?*

Theoretical Framework

This study draws on two frameworks: academic literacy and taking up space. *Academic literacy* emphasizes mathematical proficiency, practices, and discourse as interconnected components of instruction (Moschkovich, 2015). *Taking up space* centers children's agency and participation in mathematical sensemaking (Lindfors-Navarro, 2023). Both frameworks treat mathematical language as culturally and socially situated and integrated with mathematical thinking (Moschkovich, 2008, 2011). Taking up space extends academic literacy by foregrounding how children's active participation and agency in discourse enact mathematical proficiency and practices, making visible how children's participation becomes significant in constructing mathematical meaning. Together, these frameworks examine how discourse practices intersect with participation to build equitable learning opportunities within mathematics classrooms. Building on these frameworks, we designed a methodology to capture how taking up space extends academic literacy in the domain of fractions during classroom interactions.

Methods

Participants & Data Sources

We analyzed 13 one-on-one interactions from one third-grade classroom, during an equal-sharing lesson (Empson & Levi, 2011). Each student solved the same equal sharing problem: *9 children want to share 12 small cakes. How much cake can each person have if they all get the same amount?* Each interaction began when the teacher or child-initiated conversation about fraction work and ended when the teacher moved on. Cases were selected as information-rich exemplars from a classroom where the teacher demonstrated high responsiveness to children's mathematical thinking (Empson & Jacobs, 2021). The dataset included nine studens (three female, six male) in a diverse classroom. Interactions averaged 1 minute and 45 seconds in length and took place during the 30-minute circulating phase of a fraction lesson.

Analysis

Solving an equal-sharing task requires mathematical proficiency as children make key fraction problem-solving decisions signaling conceptual understanding (Hiebert et al., 2025; see Lindfors-Navarro, 2023, for details). Fractions, a challenging domain, pose distinctive academic literacy demands. These story problems require students to interpret fraction quantities, and use precise language, representations, and reasoning to communicate their thinking, making this context especially salient for examining how academic literacy is enacted and developed.

Interactions were organized into four problem-solving scenarios based on students' initial problem-solving progress (Table 1; see Lindfors-Navarro, 2023, for scenario identification). Taking a multimodal perspective, we examined video, written work and transcripts, engaging in iterative dialogue to highlight interdisciplinary insights.

Table 1

Initial Problem-Solving Progress Scenarios (N = 13)

Scenario	Description	Number
A	Valid strategy and correct answer	1
B	Valid strategy, does not yet have a correct answer	4
C	Incomplete strategy, but working towards one	3
D	Not yet completed a valid strategy and not clearly working towards one.	5

Our analysis was guided by selected questions from Gee's (2014) Discourse Analysis Building Tasks: (a) How is this piece of language being used to make certain things significant or not, and in what ways? (b) What practice(s) or activity(ies) is this piece of language being used to enact? and (c) How does this piece of language connect or disconnect things; how does it make one thing relevant or irrelevant to another? An expanded discourse analysis involving more questions was beyond the scope of this paper.

Analysis was conducted by two researchers—one with expertise is literacy and the other in mathematics education. We each viewed and coded interactions independently, then compared interpretations and negotiated coding decisions. Discrepancies were discussed until consensus was reached, ensuring analytic rigor and credibility. The mathematics education researcher clarified fraction content and its significance, while both examined how participation and multimodal actions reflected taking up space as an extension to academic literacy. To support trustworthiness, we engaged in iterative review of video data and transcripts, revisited initial codes, and documented analytic decisions throughout the process.

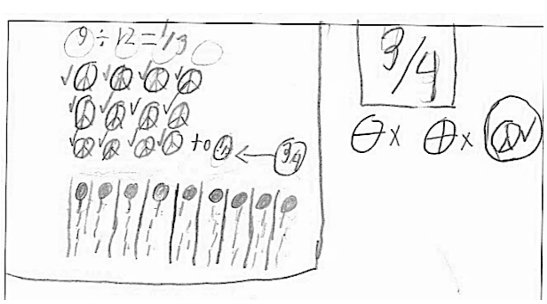
Findings

Academic literacy was evident across all scenarios and taking up space extended this literacy by highlighting how students' agency and participation shaped mathematical sensemaking. The circulating phase provided opportunities for meaningful learning, even in brief exchanges ranging from 45 seconds to about 6 minutes (M = 1.75 minutes). Some interactions involved only written work, while others included dialogue, reflecting variation in scenario complexity and depth of engagement.

Teachers supported participation by asking probing questions, encouraging students to explain their reasoning, and scaffolding the use of mathematical language; however, there were also missed opportunities when teachers accepted incomplete explanations or moved on quickly, which may have constrained deeper engagements with academic language. Most interactions began with Scenario B (n = 4) or Scenario D (n = 5). Students used direct modeling strategies, drawing pictures to model the actions or relationships (Empson & Levi, 2011). We selected examples from each scenario and consider instances that often present instructional challenges (Franke et al., 2009). All student names are pseudonyms, and the teacher is labeled as Ms. E.

Figure 1

Teddy's Written Work



Problem-Solving Scenario A: Valid Strategy and Correct Answer - Teddy

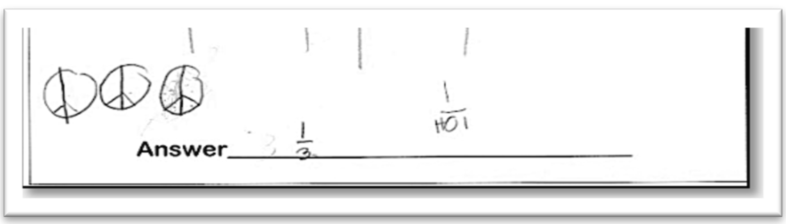
Teddy began with a valid strategy and correct answer of 4/3 but later revised it to 3/4 while explaining his reasoning aloud (see Figure 1). The teacher did not challenge the change directly but asked clarifying questions like “4 pieces of what?”, which helped Teddy maintain reasoning consistent with his original answer. This episode illustrates how symbolic shifts may reflect fragile understandings of improper fractions, even when conceptual reasoning remains intact.

Problem-Solving Scenario B: Valid Strategy, Does Not Yet Have a Correct Answer - Isaac

Isaac had a valid strategy but did not have a correct answer yet. Isaac experimented with partitioning the extra cakes into halves, fourths, and thirds (Figure 2). He chose thirds, reasoning that fourths would result in unfair distribution (Table 2, line 1). He used a valid strategy but was uncertain about naming partitions. His reasoning evolved through reflection on his written work and gestures, extending his knowledge by identifying part-whole relationships (line 1).

Figure 2

Isaac’s Written Work



The teacher’s probing questions focused on Isaac’s decision points, such as rejecting halves because they were “too big,” revealing his understanding of relative quantities (lines 1 -3). Isaac’s ability to revise and justify his reasoning, supported by responsive questioning, illustrates how conceptual understanding develops through exploration and articulation.

Table 2

Isaac Explains How He Chose to Partition the Extra Cakes into Three Pieces

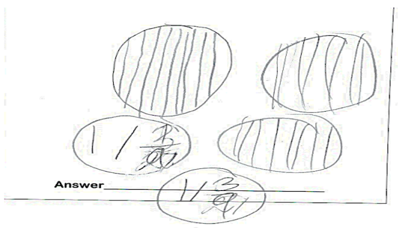
1	Isaac: So I cut them [the extra cakes] first into twos and I put them together, but I saw that it wasn't working. Next I cutted them into four but it wouldn't work because it would be too big. So I cutted them into 3. I counted...
2	Ms. E: How did you know it would be too big? [referring to Isaac’s comments that cutting into 4 pieces would be too big.]
3	Isaac: Because 4, if you go 4, 8 and then after 8 equals 12. So we'll waste all the cookie and then it wouldn't be enough for everybody.
4	Ms. E: Oh, it wouldn't be fair?
5	Isaac: Mm hmm, so I cut 'em into 3 and I counted them so like, 1 for this person, 1 for that person, 1 for this per- so the fastest way that I did it is, I knew there was 3. I just grouped them up.

Problem-Solving Scenario C: Incomplete Strategy, Working Towards One - Kade

Kade had an incomplete strategy and was working toward a valid strategy (see Figure 3). His approach revealed a partial understanding—he applied a valid strategy but initially arrived at an incorrect answer (1and 3/27). Throughout the interaction, Kade engaged in thoughtful sensemaking, particularly around two key ideas: a) identifying the size of a part relative to the whole, as evidenced by his deliberation between labeling the remaining partitions as 27ths or 9ths, and b) understanding a whole both as a single unit and as a collection of equal-sized parts.

Figure 3

Kade's Written Work for 9 Children Equally Sharing 12 Small Cakes



Kade's reasoning evolved as he reflected on his work. He reorganized his thinking, shifting from an initial answer of 1 and 3/27 to 1 and 1/9, and eventually to 1 and 3/9, although the final revision was not directly observed. His gestures, verbal explanations, and references to his drawing all supported his reasoning. The model and strategy were self-generated, indicating a strong sense of agency in the mathematical process.

The interaction was rich with key fraction decisions, and Kade's use of his drawing was instrumental in explaining his ideas. He continually returned to his visual model to justify his reasoning, especially when his initial answer proved problematic. His ability to "see" the ninths within each cake and connect that to his sharing gesture—indicating he distributed only one ninth at a time—was a pivotal moment in his conceptual clarity.

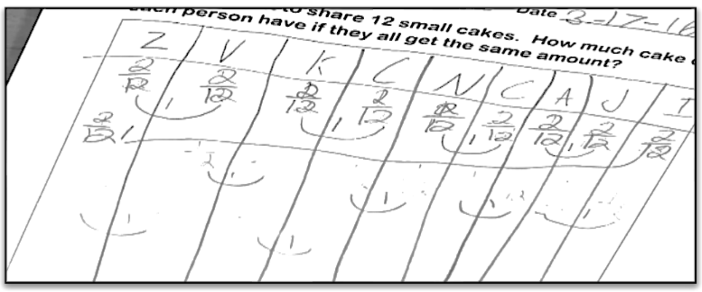
Kade's articulation of his thinking, supported by his written and visual work, enabled him to identify and correct his misconception not once, but twice. This process exemplifies how student-driven sensemaking, when paired with opportunities to reflect and revise, can lead to deeper mathematical understanding.

Problem Solving Scenario D: Not Yet Completed a Valid Strategy and Not Clearly Working Towards One - Isha

Isha did not have a valid strategy and was not clearly working toward one (see Figure 4). Isha experimented with multiple strategies, first trying 1/12, then 2/12, before deciding these "didn't really make sense” (see Table 3, line 5). This iterative process reflects productive struggle and the normalization of revising strategies. Isha mimicked the teacher’s thinking pose and used her pencil to point out twelfths, signaling metacognitive engagement (line 3). Her gestures and pointing connected to her sensemaking, showing how she reasoned about why twelfths were used and how they related to the story context (line 5). These choices demonstrate that Isha was working with reasonable quantities given the problem.

Figure 4

Isha’s Written Work



Isha raised her hand and asked the teacher for help, demonstrating persistence despite uncertainty. Her understanding was partial—she did not have an answer and was unsure of a strategy after an earlier attempt had failed. Importantly, the teacher encouraged her to try an approach that made sense after previous strategies proved ineffective.

Table 3

Isha Explains Why Her Strategy Did Not Make Sense

1	Isha: (Whispers) I don’t get it.
2	Ms. E: Don't get it. Okay, well let's go back and look at the problem. [looks at the paper] It says 9 friends want to share 12 small cakes. [Isha nods] So there's 9 people, they're gonna share the cakes. Um, so the thing is how much will each person get, right? Right? [Isha nods and looks at the teacher] So tell me what you've done so far.
3	Isha: Well, I tried 1 twelfth and it didn't equal the same amount because not each person got one. [shrugs and mimics the teacher’s thinking pose with hand on chin]
4	Ms. E: Okay.
5	Isha: And so, then I tried 2 twelfths and then that didn't really make sense for me and so..[pointed to her written work showing 2/12 repeated distributed to 9 friends]
6	Ms. E: Okay.
7	Isha: That’s why I raised my hand (1 second pause) to call you.

8 Ms. E: So, it looks like you've got two-twelfths and two-twelfths and then (1 second pause as she looks at Isha's written work) one. Would that be like (1 second pause) for one cake you're thinking?
9 Isha: (nods)

This episode highlights the complexity of problem solving. The teacher's language and actions enacted practices that normalize confusion and revision as part of mathematical thinking. By drawing attention to Isha's gestures and reasoning, the teacher supported metacognitive awareness and agency. Isha's willingness to try multiple strategies and articulate why some "didn't make sense" underscores the importance of fostering environments where exploration and adjustment are valued. Isha's engagement illustrates how conceptual understanding develops through productive struggle, reflection, and teacher support that emphasizes reasoning rather than correctness.

Discussion

Our findings extend Moschkovich's (2015) framework by showing that academic literacy is not limited to verbal exchanges but is powerfully extended through taking up space—students' agentic participation and multimodal representation in a complex domain like fractions. This study offers five key insights.

First, academic literacy was evident across all four problem-solving scenarios, even within brief interactions and taking up space amplified opportunities for sensemaking. Second, students' reasoning was dynamic and multimodal—they used speech, gestures, and written work to express and revise their mathematical thinking. These modes of expression reflect the interconnected components of academic literacy: mathematical proficiency, practice, and discourse. For example, students demonstrated mathematical proficiency through their problem-solving strategies, engaged in mathematical practices by modeling and revising their thinking, and used discourse to articulate and negotiate meaning with their teacher.

Third, teacher responsiveness was critical; probing questions supported deeper engagement, while missed opportunities sometimes constrained students' ability to take up space. Fourth, students demonstrated agency in their sensemaking, as seen when Kade and Isaac independently revised strategies—illustrating how taking up space operationalizes academic literacy in practice. Finally, when children exhibited fragile understandings, such as Isha's struggle with twelfths, these moments became openings for learning rather than constraints. We recommend future research explore circulating phase interactions with other problem types and investigate how partial understandings and agentic participation shape mathematical growth.

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COLLEGE STUDENTS' PERCEPTIONS OF ASSISTED ORAL EXAM FORMATS

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To provide students with more and better opportunities to demonstrate their knowledge of mathematics, instructors create assisted oral exams. In these exams, students meet with their instructor in one-on-one settings to discuss how students can apply their learned knowledge in new settings, prove previous results, or reflect on what they have learned. This report covers how students perceive their learning through taking assisted oral exams in two different upper-division undergraduate mathematics courses taken in the same semester. Students identified different strengths and weaknesses of the assisted oral exam formats and the opportunity to learn new material in both classes.

Assessments beginning at the earliest ages and continuing through to university instruction, are changing in perception, from a strict gatekeeping tool to a dynamic instrument to measure current progress, provide timely feedback, and deepen student knowledge. While the history of assessments is for the grading and ordering of student performance (Shavelson, 2007), mathematics instructors at all levels want to implement assessments that are faithful to the material presented and representative of the knowledge students have or gained through participation in the class. Our focus will be on how individual students perceive developing a deeper knowledge through performing a particular format of assessment. The research question guiding this study is: *How do undergraduate mathematics majors perceive their own expectations of mathematical growth through different assisted oral examination processes?*

Literature Review

Assisted oral exams have been part of undergraduate mathematics courses in the United States during the 21st century. The work of Carter et al. (2016) illustrates how the exams help students build confidence in their knowledge of calculus at Seattle University. Their work treated assisted oral exams as a formative assessment, a tool to give instructors data on what the students know about the content without receiving a grade or other formal feedback. In the United Kingdon, work by Iannone et al. (2020) illustrate tensions students face between traditional, pencil-and-paper, closed examinations with the oral examination format. Many participants expressed comfort with traditional exam format due to familiarity; however, this did not make the experience enjoyable or indicative of their learning. The format, from the students' perspective, only required a rudimentary regurgitation of material to be written quickly in books, given time constraints that instructors put on closed-book examinations.

In the early volumes of *Investigations in Mathematics Learning*, Soto-Johnson and Fuller (2012) showed some students courses either could provide a written proof of a concept from the abstract algebra course but not verbalize a similar explanation, or the opposite: some students could explain verbally how to show a concept from abstract algebra is true but could not provide a written proof of the same concept. A similar result also appeared in Theobold's (2021) work with students enrolled in statistics classes, where students who might have struggled to communicate knowledge of p-values on written examinations could accurately and confidently communicate knowledge of the material by discussing with the instruction through conversation.

We used principles from Deep Approaches to Learning by Anderson et al. (2016) to provide a framework to ground students' reflections. A list of those 12 principles can be found in Table 1.

Table 1

Deep Approaches to Learning (Anderson et al., 2016, pp. 7–8)

Label	Statement
HO1	Analyzing the basic elements of an idea, experience, or theory, such as examining a particular case or situation in depth and considering its components.
HO2	Synthesizing and organizing ideas, information, or experiences into new, more complex interpretations and relationships.
HO3	Making judgments about the value of information, arguments, or methods.
HO4	Applying theories or concepts to practical problems or in new situations.
R1	Examine the strengths and weaknesses of your own views on a topic or issue.
R2	Try to better understand someone else's views by imagining how an issue looks from their perspective.
R3	Learned something that changed the way you understand an issue or concept.
I1	Worked on a paper or project that required integrating ideas or information from various sources.
I2	Included diverse perspectives in class discussions or writing assignments
I3	Put together ideas or concepts from different courses when completing assignments or during class discussions
I4	Discuss ideas from your readings or classes with faculty members outside of class
I5	Discuss ideas from your readings or classes with others outside of class

Key: HO = Higher Order Learning R = Reflective Learning, I = Integrative Learning

Method

This study utilizes interpretivist phenomenology to guide data collection and analysis. Bringing in McGovern's (2017) conceptualization from the study of health care and incorporating Seherrie's (2024) understanding of this methodology, we hope to capture experiences of undergraduate mathematics major performing a new-to-them assessment experience, assisted oral exams. We interpreted participants' own sense-making of their cognitive and affective states during the assessment. Validity is established through a two-round coding process: an initial pass of inductive open coding to capture emergent themes followed by a deductive pass utilizing the NSSE Deep Learning framework to ground these experiences in established educational theory.

Participants in the study are students pursuing a major in mathematics in their third or fourth year of undergraduate study. Eight students each completed an online survey that members of the research team created by adapted from statements in Table 1. Selected statements from the survey can be found in Table 2. After completing the survey, one member of the research team invited the participants to participate in a semi-structured interview, found in Table 2.

Table 2

Semi-structured Interview Guide

Number	Interview Question
1	Tell us about your experiences with the Assisted Oral Exam. What do you think went well?
2	One of the survey statements asked you to what extent did you agree with the statement, "The assisted oral exam was helpful in attaining a deeper understanding of difficult mathematical concepts." You selected [survey response]. Why's that?
3	One of the survey statements asked you to what extent did you agree with the statement, "The assisted oral exam was helpful in improving your ability to communicate difficult mathematical concepts." You selected [survey response]. Why's that?
4	One of the survey statements asked you to what extent did you agree with the statement, "The assisted oral exam was helpful in relieving anxiety related to working difficult math problems." You selected [survey response]. Why's that?
5	On the survey, you wrote about how this exam contributed to your deeper understanding. You wrote, [survey response]. Can you tell me more about what you wrote?
6	Also on the same survey, you wrote about preparing for this exam affecting your confidence in communicating mathematics. You wrote, [survey response]. Can you tell me more about what you wrote?

Some of the interviewed participants had been simultaneously enrolled in two courses with assisted oral exams in the same semester. Five of the eight participants enrolled in both courses that served as the settings for this study for this semester took part in this study. In this report, we will look at commonalities and differences between participants expressed across the two courses. In examining the similarities and differences, researchers read the interviews and coded the interview data thematically using the Deep Learning statements (Anderson et al., 2016). We believe that the students' responses match the Deep Learning statements when three researchers applied the same code to one student's response for one interview question relating to one of the two classes. In our analysis, researchers reached consensus on the statement "[e]xamined the strengths and weaknesses of your own views on a topic or issue" (Anderson et al., 2016, p. 8).

Findings

The level of agreement for the survey statements can be found in Table 3. Each of the five participants who enrolled in both courses are listed here, a single letter used for pseudonyms. We provide additional student feedback on two of the six statements: one statement where all participants agreed with the provided statement and one where the participants did not express full agreement with the statement provided. The participants agreed on half of the statements for Abstract Algebra but only agreed on one statement when reflecting on Combinatorics.

Table 3

Participants' Agreement of Survey Statements, Separated by Course Taken

Participant	Abstract Algebra						Combinatorics					
	1	2	3	4	5	6	1	2	3	4	5	6
A	A	a	A	A	a	a	n	n	a	n	n	a
B	A	A	A	A	a	a	n	a	a	a	A	A
C	A	a	A	A	n	n	a	A	A	a	a	a
D	a	a	a	d	D	D	d	d	a	a	n	n
E	A	a	A	a	d	d	a	A	a	a	A	a

D = strongly disagrees, d = disagrees, n = neither agree or disagrees, a = somewhat agrees, A = strongly agrees.

Consistent Agreement: Communicating Difficult Mathematics

The third statement in the survey is found in the third question of the semi-structured interview in Table 2. In the open-ended question portion of the survey, Participant B wrote about their “being given an environment to show off what I’ve learned…at my own pace made me feel more comfortable that math is for me” for Abstract Algebra. Additionally, Participant E wrote that the format “helped me communicate ideas about how I would think about solving a new problem” for the same course. Participant A, writing about Combinatorics, wrote that “the conversation…helped me talk through what I know and when I reached the end of that [the instructor] could push me toward what I was missing”. Also in Combinatorics, Participant E wrote about how preparing “a mini lesson to present helped me in my confidence in presenting ideas that I don’t have full knowledge of”. Participants elaborated on their agreement regarding learning through communication in interviews following the completion of the survey. Responses from the participants can be seen in Table 4.

Table 4

Quotes from Participants Regarding Communicating Difficult Mathematics

Participant and Course	Quote from Interview
A (Abstract Algebra)	It’s not that I’m trying to prove every, get into the nitty gritty of all the details behind how it’s working, but it’s like we have a problem and I’m saying OK, with this problem we can approach it this way and we can apply this theorem and that will lead us to a solution. And if I, you know, took the time to write it all out...it would create a nice proof. But the writing it all out isn't what we’re working on. It’s the intuition behind it.
B (Abstract Algebra)	So it helped not just in how I could communicate with...about like what I had learned, but it also helped with them being able to take that and communicate it to like my parents. So it helps with, you know, being able to communicate, like, bigger picture ideas.
C (Abstract Algebra)	Specifically, in these higher math classes...you could regurgitate a definition, but do you actually know what it’s saying? Or the further applications?
D (Abstract Algebra)	In addition to sort of saying those topics, you really had to provide more detail and more explanation as to what you knew”
E (Abstract Algebra)	Come up with ideas on the spot and being able to like have the knowledge to articulate them... "It almost felt like I was the teacher... oral communication to talk through my thought processes."

No consensus in agreement: Contributing to Relieving Anxiety

The fifth statement in the survey is found in the fourth question of the semi-structured interview in Table 2. When asked about their agreement to assisted oral exams contributing to relieving anxiety related to mathematics, participants expressed the full range of agreement choices, with participant strongly agreeing to the statement for Combinatorics to two selections of disagreement, both in Abstract Algebra. Although the participants did not elaborate on the relief of anxiety in the survey, they did elaborate on their experiences of feeling the relief of anxiety during the assisted oral exams in the interview portion of this study. Responses from the participants can be found in Table 5.

Table 5

Notable Quotes from Participants Regarding Relieving Anxiety

Participant and Course	Quote from Interview
B (Combinatorics)	I kinda don't know what I'm doing, then that's OK....I think that [the instructor] creates a really good space for talking about math and I can tell that [the instructor] really just wants to make sure that we're learning and that that's always [the instructor’s] main priority
D (Abstract Algebra)	Just because it was like not like any other oral exam I've done. So I felt the least prepared for it...Like for the practice, we were required to pick 3 topics and then do three questions for each of those with like a level 1, 2 and 3 difficulties....And then when I got to like the actual exam, it wasn't structured like that....We spent the whole time on 1 main topic and then it was mostly middle difficulty
E (Abstract Algebra)	I really wasn't sure what exactly was going to happen or, if maybe I was most worried about being put on the spot and not knowing what to say.... I also didn't feel like I did well...Like what I was scared of didn't actually happen in the test...I guess there are also times because it was a new question and I was still focused on what I would be graded on. I wasn't exactly sure if the new question was stuff that I was expected to know how to fully do...I mean, I know a little bit like I know the steps of how I would solve it, but I can't solve it for you right now. So that was sort of scary or gave me some anxiety, I guess.

Discussion

All five participants who took both courses simultaneously agreed that the assisted oral exams supported their own ability to communicate difficult ideas across both courses. Communication, going back to the early standards documents of the National Council of Teachers of Mathematics (1991) and continuing through the *Common Core State Standards* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), is a key activity for students to convince themselves and their teachers they have built a rich, conceptual understanding of the mathematical material presented in class. By students seeing the importance of communicating mathematics, both at the university level and years after the publication of these documents, we feel that we have achieved a goal of having students recognize they are taking ideas and content from their classes and apply them to telling another what they know and what they learned.

All participants expressed sentiments in their interviews consistent with the first Deep Learning indicator regarding reflective learning (Anderson et al., 2016), as determined by the coders. Participants in this study expressed a variety of evaluative statements, ranging from "Oh, I'm so dying to talk about groups and rings" (Participant A) to "there's always that anxiety of like, what if I forget the easiest thing I know" (Participant C) when reflecting on their experience taking an Assisted Oral Exam in Abstract Algebra. Because the Abstract Algebra Assisted Oral Exam used in this study, is a test-like format of the exam for the course, instead of the discussion-based format of the Combinatorics Assisted Oral Exam in this study, students may be more aware of their learning in the Abstract Algebra class rather than Combinatorics. Prior research indicates increased student learning when students have opportunities to reflect on what they have learned and are learning, evaluating their own learning processes (Hoffman et al., 2024; Lee & Ma, 2019).

One implication for practice is a reflection on the kinds of questions instructors can ask students in Assisted Oral Exams. As reported here, some questions encourage students to reflect on what they learned, while other questions encourage on how they learned. Students see value in both question formats, particularly in communicating difficult mathematical material to others. While most students have likely practiced communication of mathematics in school settings (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010), Assisted Oral Exams allow students to continue that valuable practice to the collegiate mathematics classroom. We are collecting additional data to determine if students share the same sentiments in other mathematics courses, not only for courses for students pursuing degrees and majors in STEM fields, but for students enrolled in mathematics courses beyond those in the STEM fields, such as content courses for future elementary teachers, building upon the work of Visscher and White (2015).

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Mathematical Reasoning, Creativity, and Sense-Making

SECONDARY STUDENTS' REASONING ABOUT FINDING THE DISTANCE BETWEEN TWO POINTS

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This study collects secondary students' strategies for finding the distance between two points after regular classroom instruction. Fifteen students participated in clinical interviews containing questions about finding the distance between two points on a coordinate plane. The most common reasoning strategies used by students included "Count Diagonally," "Slope," "Count 1-Dimension," "Rotate," and "Count Legs," and "Pythagorean Theorem." This study contributes understanding of how secondary students reason about linear measurement in the context of finding the distance between two points.

Introduction

Student outcomes are alarmingly low on geometric measurement tasks (Battista, 2012). Research suggests that students' measurement reasoning is shallow as a result of enacting ascribed formulas before developing a conceptual understanding of those formulas (Battista, 2012), rather than progressing through appropriate levels of understanding (Barrett & Battista, 2014; Clements & Sarama, 2021). For example, once students have used the more procedural distance formula, they no longer apply the Pythagorean Theorem (PT) approach.

Measurement reasoning includes determining, reasoning about, and operating on measurements, and is critical for conceptual understanding of shapes, transformations, and the coordinate system (Battista, 2012). Researchers have consistently found that early measurement reasoning is likely to include non-numerical comparisons, followed by making connections to numerical procedures using rulers or formulas, and finally demonstrating relationships between measurement and formulas, such as perimeter (Barrett & Battista, 2014; Battista, 2012; Clements & Sarama, 2021). At the highest levels, students use the PT to find the distance between two points without prompting and in unfamiliar contexts (Battista, 2012). While these levels provide detailed descriptions of students' mental operations at each level, and how instruction might advance students' reasoning from one level to the next, they are primarily focused on horizontal and vertical distances only. Battista's (2012) levels, for instance, do not consider the introduction of diagonal distances in a coordinate plane until the most sophisticated level (level 7). Thus, there is no inclusion of variability in the ways that students might reason about diagonal distances between two points in a coordinate plane. The present study begins to address that gap.

Mastery of the PT is a gateway to success in higher mathematics. For instance, Tallman et al. (2021) determined that 47.7% of related rate problems in calculus utilized varying quantities that could only be modeled by the PT (Tallman et al., 2021), making it critical that students are prepared to apply to the PT upon exiting high school.

Far less memorable than the PT is the Distance Formula (DF) (Asdourian, 2012; Bartlett, 2022; Carter, 2024). Although math education research has not thoroughly examined students' unbalanced recollection of the PT over the DF, it could be explained by instruction that jumps too quickly to teaching the procedural DF without building a solid connection to the more conceptual PT (Battista, 2008; Empson & Levi, 2011; NCTM, 2023). Therefore, this study asked: *How do secondary students' reason about the distance between two points on a coordinate plane after regular classroom instruction?*

Methods

Fifteen students in grades 8-11 who were willing to participate in three interviews and received parental permission were selected for this qualitative interview study. The students were enrolled in Algebra I, Geometry, or Algebra II (Table 1) and had experienced instruction about the PT to find the distance between two points in a coordinate plane. Additionally, students in Algebra II would have experienced instruction about using DF during their previous Geometry class. Geometry students may have experienced similar instruction. Algebra I students had not received instruction connecting the PT to the DF.

Table 1

Participant Information

Math Class	Number of Students	Participants
Algebra I	6	01, 03, 07, 08, 10 14
Geometry	4	06, 12, 13, 15
Algebra II	5	02, 04, 09, 11, 16

Research Design and Data Collection

Two tasks were presented on paper during the interview. The first task stated "The vertices of a parallelogram are (7, 13), (13, 5), (10, 1), and (4, 9). Find the perimeter of the parallelogram" (Battista, 2012). The task captured whether students' strategy to find the distance between two consecutive vertices. The second task asked students to find the distance from home to school, where home and school were shown as two points on a coordinate plane. It was anticipated that participants would solve the task using PT, DF, or in non-normative ways (Battista, 2012). Interviews were conducted at informal locations (e.g., YMCA, participants' home). Interviews were recorded and transcribed. All written work was collected. Notes were added to the transcript to indicate participants' non-verbal actions. Transcripts were coded inductively by independently by two researchers for students' reasoning strategies to find distance.

Findings

We summarize findings addressing the research question: how do secondary students reason about the distance between two points after regular classroom instruction? We provide samples of student work that typify each reasoning strategy that was identified. All 15 participants counted something as part of their reasoning strategy, although they differed in what they counted and how they used that information. Some students tried more than one strategy. The strategies are summarized in Table 2. We note that the strategies are not leveled, as many students moved between strategies during their interviews.

Table 2

Summary of Results

	Algebra 1						Geometry				Algebra 2				
	1	3	7	8	10	14	6	12	13	15	2	4	9	11	16
Count Diagonal	X	X	X	X	X	X	X		X	X					
Slope	X				X		X				X		X	X	X
Count 1-Dimension	X						X			X	X		X	X	
Rotate	X	X	X		X						X				X
Count Legs					X		X		X		X		X		X
Count L-Shape and Z-Shape				**C**		X									
Eyeball			X								X				
Pythagorean Theorem								**C**				**C**	**C**		
Distance Formula			X							X					

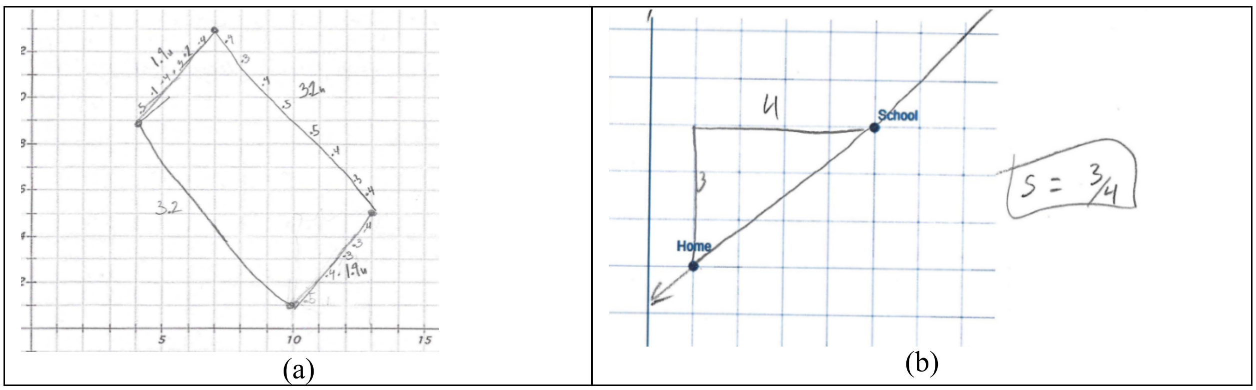
Note. X indicates the student used a strategy and did not arrive at a correct answer. A bold C indicates the student used a strategy and arrived at a correct answer.

Count Diagonal

The most common strategy was count diagonally (Participants 01, 03, 06, 07, 08, 10, 13, 14, 15). Participants who reasoned this way counted each diagonal unit as one or a fraction/decimal of one (Figure 1a). Participant 03 recognized that the diagonal sides of the parallelogram were not aligned with the horizontal and vertical units on the graph and expressed that the length should not be counted with whole numbers. She estimated each piece as a decimal. None of the nine participants who counted diagonally reached the correct answer with this strategy nor did any of their work lead to a different strategy to find the correct answer.

Figure 1

Examples of Count Diagonal (left) and Slope (right)



Slope

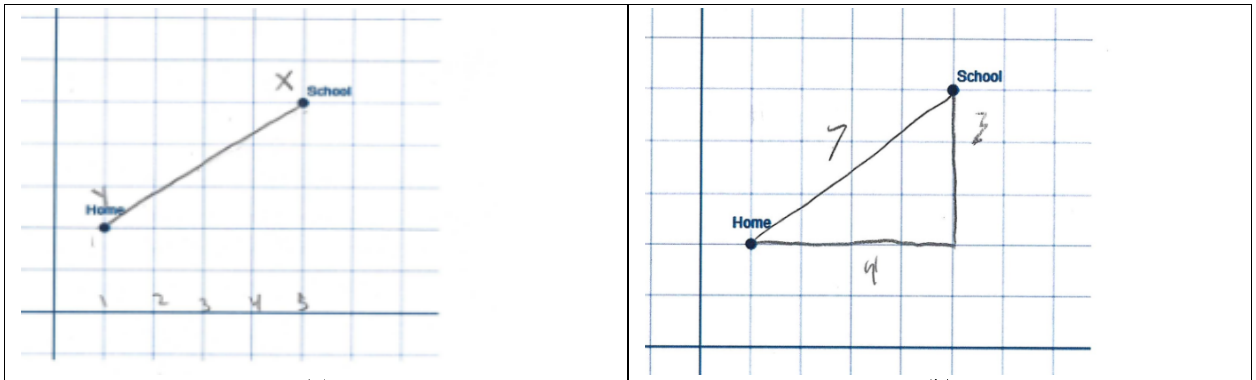
Seven participants (01, 02, 06, 09, 10, 11, 16) counted the horizontal and vertical shifts and calculated the slope of the line segment rather than finding the distance between the points (Figure 1b). They each recognized the answer did not make sense. Only Participant 09 (1 participant out of 7 = 14.3%) went on to find the correct distance. He shared his discomfort with the fact that his answer of three-fourths (slope) on both tasks was a fraction, which did not fit his understanding of distance. As he struggled, he realized that he had found the legs of the right triangle, and recognized the relationship fit the PT. The other six participants reached the answer of three-fourths (slope) and either accepted it or explained that they were not sure how to adjust.

Count 1-Dimension

Six participants (01, 02, 06, 09, 11, 15) counted in 1-dimension. Participants who reasoned in this way attended either the horizontal or the vertical shift, but not both (Figure 2a). Five participants (01, 02, 06, 13, 15) counted the horizontal shift and one participant (09) counted the vertical shift. All the participants recognized that counting 1-dimension resulted in an estimate. Participant 15 called it a “very rough estimate.” None of the participants reached a correct solution by counting in 1-dimension. However, Participant 09 (1 out of 6 = 16.7%) eventually progressed to the PT and found the correct answer.

Figure 2

Examples of Count 1-Dimension (left) and Count Legs (right)



(a) (b)

Rotate

Rotate describes when students attempted to rotate the diagonal distance to be oriented horizontally. Students either mentally rotated the segment or used their fingers as a measure. After rotating, students counted using the length of the horizontal segment that was created by rotating. All six participants who used this reasoning strategy (01, 02, 03, 07, 10, 16) gave verbal explanations and used gestures to demonstrate. On the first task, Participant 16 estimated the exact distances for the two sides and so calculated the perimeter correctly. The other five participants did not reach the correct solution by rotation.

Count Legs

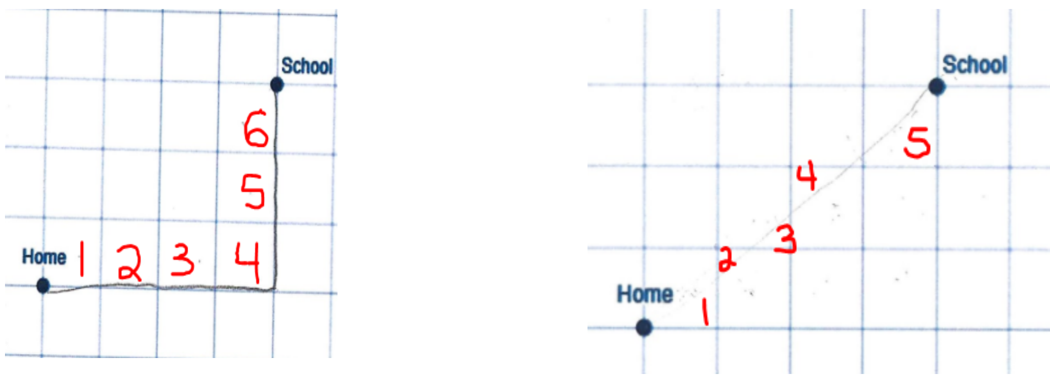
Six participants counted the horizontal and vertical shift but expressed uncertainty about how to use the values to find the distance of the diagonal segment (Figure 2b). These participants (02, 06, 09, 10, 13, 16) recognized the importance of the shift values, and some even mentioned the PT and DF, but they did not use the values to find distance. Of the participants who counted legs, only Participant 09 (1 out of 6 = 16.7%) went on to find the correct answer. The other participants added (10), multiplied (13), doubled the line (02), or expressed uncertainty about how to proceed with the values (06, 16).

Count L-Shape and Count Z-Shape

Two Participants (08, 14) counted boxes using both an L-Shape and a Z-Shape (Figure 3). When counting in an L-Shape, participants counted the boxes in the horizontal and vertical direction around the outside of a triangle, counting the box in the right vertex only once. When counting Z-Shape, participants counted the boxes in a stair step pattern running down the line and bouncing back and forth on either side of the line. Both participants started task 2 by counting an L-Shape and moved to a Z-Shape when asked if they could go straight between home and school. Neither participant reached the correct answer by using Count L-Shape. Participants 08 reached the correct answer using the strategy Count Z-Shape, although she was unable to repeat the success on other similar problems. These reasoning strategies did not lead either participant to a conceptually appropriate method.

Figure 3

Examples of Count L-Shape (left) and Count Z-Shape (right)



(a) (b)

Note. Red numbers were digitally inserted by the researcher to indicate students' counting pattern.

Eyeballing

Two participants (02, 07) estimated the distance between two points based on the look of the graphed line. Although their estimates were reasonable, the eyeballing strategy did not lead to a correct answer, and neither developed a reliable reasoning strategy to find the correct answer.

Pythagorean Theorem

Of the 15 participants, two (04, 12) used the PT as their first reasoning strategy on both tasks, and one participant (09) used it on the second task after exploring other strategies. All three participants arrived at the reasoning strategy independently and found the correct answer.

Attempts at the Distance Formula

Two participants (07, 15) referred to the DF. Participant 07 said, "Then you do something, and you do another thing over here, then some other operation, then it equals that distance y." Participant 15 said, "Well, usually you have like a little math problem over here (point at school) and the little math problem over here (point at home) and you like subtract, or divide." Neither student was able to enact the distance formula and use it to solve the task.

Discussion

Students' reasoning strategies included several strategies that were not consistent with accurate measurement reasoning about the distance between two points. This finding aligns with Battista's (2012) report that student outcomes in measurement reasoning are alarmingly low due to weak conceptual understanding. Additionally, similarities in student reasoning strategies were notable, which suggests that these types of erroneous measurement reasoning are not isolated. These findings could be attributed to an instructional focus on the DF or PT as a procedure rather than as interconnected concepts which prevents the development of critical foundational understanding and leads students to struggle with using the procedure (Battista, 2012; NCTM, 2023). Rather, students who focus on the concept are better prepared when they encounter the concept again (Battista, 2008: NCTM, 2023), as indicated in this study by the students who applied the PT correctly.

The results of the current study provide a basis for understanding how high school students reason about finding the distance between two points on a coordinate plane, which extends prior levels of measurement reasoning (e.g., Barret & Battista, 2014; Battista, 2012; Clements & Sarama, 2021). Previous research that has identified levels that begin exclusively with ways that student's reason about horizontal and vertical distances, and do not adequately address measurement reasoning about the PT and coordinate systems until their most sophisticated level (e.g., Battista, 2012 includes this type of reasoning only in level 7, and no level 8 exists). This research contributes that even among high school students who reasoned normatively about horizontal and vertical distances, their preparedness to reason about distances in a coordinate plane using the PT was varied, and for many students, limited.

The largest issue raised is our finding that students' strategies for finding the distance between two points were rarely based in conceptual facts, with only 3 out of 15 able to complete the task correctly after regular classroom instruction on the topic. This result raises questions about the relationship between instruction and long-term conceptual learning. These questions require further examination, including effective instructional components that intercept misconceptions and lead to conceptual understanding.

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FOSTERING MATHEMATICAL CREATIVITY IN PRE-SERVICE TEACHERS THROUGH PROBLEM-POSING AND DIVERGENT THINKING

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This study investigated the connection between mathematical creativity, problem-posing, and divergent thinking in university-level pre-service secondary mathematics teachers. The goal was to examine if purposeful instruction integrating problem-posing and divergent-thinking tasks could foster mathematical creativity. Throughout a 16-week required course, participants engaged in these teaching methods three days each week. Mathematical creativity was measured via pre- and post-tests and three sets of Divergent-Thinking Exploration Questions. The analysis revealed the possible effects of these teaching methods on the participants' mathematical creativity and the potential impact on K-16 mathematics curriculum.

Creativity remains a highly debated, context-dependent concept without a universal definition. While artistic products are readily deemed creative, defining mathematical creativity involves the ability to discern patterns, engage in non-algorithmic decision-making, and produce something new (Yuan & Sriraman, 2011). Too often, mathematics is mistakenly viewed as the antithesis of creativity, focusing only on rules and analysis, thereby limiting exploration.

Torrance (1988) defined creativity as sensing differences, gaps, or missing elements in a situation. Sriraman and colleagues (2011) further highlighted three key components: fluency, flexibility, and originality. To foster this, educational settings need to offer meaningful, open-ended mathematics experiences. Current literature suggests that multiple representations and Sriraman's five principles (Gestalt, Aesthetic, Free Market, Scholarly, and Uncertainty) can enhance mathematical thinking and creativity (Kiymaz et al., 2011). Challenging students with divergent thinking tasks can also encourage multiple responses and innovation (Chamberlin & Weiss, 2023). At its core, creativity pushes students to innovate, make new connections, and create something new from existing material (Sedivy-Benton et al., 2014). This leads to the central inquiry: can mathematical creativity be fostered and developed in the K-16 classroom?

The research question is: *How can mathematical creativity scores of university-level mathematics majors be positively impacted (when comparing their pre-and post-test) by treating them with regiments of divergent thinking and problem-posing and alter future K-16 mathematics curriculum?* This study addresses a gap in resources regarding mathematical creativity, which is key to improving students' problem-solving skills, an objective outlined by the National Council of Teachers of Mathematics (NCTM, 2000). Problem-posing and divergent thinking are considered primary forces for encouraging creativity (Toledo & Vistro-Yu, 2022).

Divergent thinking is defined as the generation of varied output from the same source of information, emphasizing multiple strategies and numerous ideas through open-ended questions (Yuan & Sriraman, 2011). Problem-posing is the process by which students construct new mathematical problems based on their experiences and alterations to concrete situations (Brown & Walter, 2005). The most effective problem-posing involves a sequence of steps including problem identification, alteration, and generalization. Since problem-posing allows students to examine and create their own problems, it is considered an effective method for developing the relationship between mathematical knowledge, skills, and process (NCTM, 1991; Singer et al., 2013). Empirical data suggests that exposure to creativity-enhancing activities, like problem-posing and divergent thinking, strengthens mathematical creativity (Sriraman et al., 2011).

Theoretical Framework and Literature Review

The core challenge in K-16 mathematical education is finding effective methods to foster problem-solving and divergent thinking. Since creativity is an essential component of problem-solving, increasing mathematical creativity is hypothesized to enhance these skills. Creativity is a complex, often controversial subject without a single, universally accepted definition. Historically, it was described as the ability to find patterns and engage in non-algorithmic decision-making (Yuan & Sriraman, 2011). Modern interpretations often emphasize utility and novelty (Sternberg & Lubart, 1999), or the ability to produce original work that significantly extends the body of knowledge (Juter & Sriraman, 2011).

K-16 Mathematical Creativity

Mathematical creativity is often described as a product, a process, or a combination of both. Torrance defined it as sensing differences, gaps, or something ‘not right’ in a problem, which can lead to educated guesses and hypothesis formulation (Yuan & Sriraman, 2011). Sternberg agreed, calling it the “ability to produce unexpected work [deemed useful and adaptive]” (Juter & Sriraman, 2011). These definitions share a common thread: exposure to uncertainty. In the K-12 setting, mathematical creativity is defined as the process leading to unusual and/or insightful solutions to a problem, the formulation of new questions, or the ability to view an old problem from a new angle (Fetterly & Wood, 2018). While K-16 students may not create new mathematics, they can rediscover ways to solve older problems.

Kaufman and Beghetto’s (2009) Four C Model of Creativity provides a framework for understanding creative acts in the classroom. Mini-C refers to interpretive creativity, such as a student interpreting a scenario differently. Little-C is everyday creativity, which involves combining old knowledge with new information. Pro-C is expert creativity, exemplified by developing a new instructional technique. Finally, Big-C represents legendary creativity, like the discovery of a mathematical theorem. Teachers can cultivate these levels of creativity by choosing specific problems that encourage creative thought (Sedivy-Benton et al., 2014).

Encouraging Creativity

Mathematical creativity is often described by the theoretical framework of Sriraman's five principles: Gestalt (incubation and insight), Aesthetic (combining ideas for atypical approaches), Free Market (taking risks and defending ideas), Scholarly (contributing to and respectfully debating the body of knowledge), and Uncertainty (residing in the unknown nature of real-world problems) (Juter & Sriraman, 2011). Encouraging this creativity in the K-16 classroom involves presenting challenging tasks to elicit multiple responses and building activities which force individuals to bend and think outside the proverbial box (Sedivy-Benton et al., 2014).

Evaluating Creativity: Fluency, Flexibility, and Originality

To quantify mathematical creativity, the operational skills of fluency, flexibility, and originality are adopted (Kiymaz et al., 2011). Fluency is the flow or number of relevant ideas a student can produce. It requires recalling previous knowledge to generate as many relevant responses as possible. Flexibility is the number of different categories or approaches covered within the responses. This is key in mathematics since not all problems can be solved the same way and require transitioning between different types of thinking. Originality is the statistical rarity or "production of unusual, far-fetched, remote, or clever responses" (Yuan & Sriraman, 2011). These three skills are central to the psychometric approach to analyzing creativity, which quantifies the concept using paper-and-pencil tasks like Balka’s creativity test (1974).

The Role of Problem-Posing and Divergent Thinking

Problem-posing is an essential link to mathematical creativity (Toledo & Vistro-Yu, 2022). It is defined as the process where students construct their own meaningful mathematical problems based on interpreting concrete situations (Yuan & Sriraman, 2011). Problem-posing can be free (generating a problem from a general prompt), semi-structured (completing a partially open situation), or structured (modifying a specific, given problem).

The process of problem-posing involves problem identification, problem attribution, problem alteration, alteration exploration, and alteration generalization (Brown & Walter, 2005). By altering problem characteristics, students engage in a creative, dynamic process that utilizes previous knowledge and fosters mathematical creativity (Wilkerson et al., 2015). Studies show a significant relationship between creativity and problem-posing, especially when students have greater content knowledge (Yuan & Sriraman, 2011). Divergent thinking is a critical skill for problem-posing and is defined as the "generation of information from given information, where the emphasis is on a variety of output from the same source" (Yuan & Sriraman, 2011, p. 51). Divergent thinkers typically respond using multiple modes of representation, and the tasks naturally motivate the development of fluency and flexibility (Chamberlin & Weiss, 2023).

Methods

Participants and Procedure

This study was designed as a one-group pre-post test design. The sample was comprised of six junior and senior students enrolled in a required 16-week "Functions and Modeling" course at a public university. Research instruments were integrated as graded course assignments, though work was only analyzed for consenting students. Participants received purposeful problem-posing and divergent-thinking treatments for 50 minutes, three times a week. This included both instructor-led exposure and relevant homework assignments requiring multiple solution methods.

An example of the scenario is that some people say that to add four consecutive numbers, you add the first and the last numbers and multiply the result by 2. What can you find out about that? After this situation is investigated, problem posing would begin by asking other questions that change one or more attributes of the original problem. The -problem posing- could then look like the following: Some people say that to add six consecutive numbers, you add the first and the last numbers and multiply the result by X. What if there were eight consecutive numbers? This process could then continue to formulate a generalization.

One way to express divergent thinking is to ask students to solve a problem in more than one way. For example, give the students this task: use square tiles to surround a 1×1, 2×2, 3×3, …, $n\times n$ square. Find the total number of tiles needed to surround the square. There are many equations students may produce (for instance, $T=n+n+n+n+4$; $T=4(n+1)$; $T=4n+4$; $T=2n+2(n+2)$; $T=8+4(n-1)$; $T=(n+2)^2-n^2$; $T=n^2-(n-2)^2+8$). Exposing students to these various equations allows them to experience divergent thinking in mathematics.

Research Instruments

Balka's Creative Mathematical Ability Test (CMAT): The CMAT (1974) served as the primary pre- and post-test instrument, utilizing open-ended problem-solving and problem-posing questions designed to elicit divergent responses. This instrument is scored operationally across three distinct criteria. Fluency is measured by the total number of relevant answers given, emphasizing "quantity over quality," with each relevant answer earning one point. Flexibility is assessed by counting the number of unique categories or "trains of thought" represented in the answers, where each unique category earns one point. Originality measures the uncommonness of a response relative to the entire group: responses appearing less than 2% across all students earned two points, those appearing 4% earned one point, and those appearing 6% or greater earned zero points. The CMAT was administered without resources outside of class at the beginning and end of the course, and the same six items were scored by the researcher simultaneously across all participants for consistency.

Divergent Thinking Exploration Questions (DTEQ): The DTEQ, adapted from Haylock (1987), consisted of three sets of five questions administered throughout the semester to monitor growth in a specific aspect of creativity: fluency. Questions explored categories such as mathematical operations, variable values, creating shapes, and finding similarities between functions and figures. Scoring for the DTEQ sets was based solely on the Fluency section of Sheffield's Depth and Creativity Rubric (DCR) (2000). DCR defines fluency as the number of different and correct answers, solutions, or questions raised. For this project, each relevant "thought" or response presented by the student earned one point, allowing the researcher to track progress in idea generation (fluency) during the treatment phase.

Findings

Creative Mathematical Ability Test (CMAT) Results

The CMAT, which measures overall creativity, fluency, flexibility, and originality, showed a clear increase in scores across the group after the 16-week treatment (Table 1). The greatest numerical gain was observed in fluency (the quantity of ideas), followed by flexibility (the number of different idea categories).

Individual CMAT Score Increases

Individual student data also supported the overall trend, showing score increases across all creativity components:

- Student A: Increased by 18 points overall (Fluency: +11, Flexibility: +6, Originality: +1).
- Student B: Increased by 25 points overall (Fluency: +10, Flexibility: +8, Originality: +7).
- Student C: Increased by 19 points overall (Fluency: +7, Flexibility: +7, Originality: +5).
- Student D: Increased by 17 points overall (Fluency: +6, Flexibility: +3, Originality: +6).
- Student E: Increased by 8 points overall (Fluency: +2, Flexibility: +2, Originality: +4).

These consistent gains suggest promising trends that purposeful problem-posing and divergent thinking methods positively impact mathematical creativity.

Table 1

Average Scores on the Creative Mathematical Ability Test (CMAT) Pre- and Post-Test

Category	Pre-Test Average	Post-Test Average	Increase (Points)
Overall Creativity	55.60	60.83	5.23
Fluency	20.80	23.33	2.83
Flexibility	16.20	17.83	1.63
Originality	18.60	19.67	1.07

Divergent Thinking Exploration Question (DTEQ) Results

Three sets of DTEQs were administered throughout the semester to specifically monitor the growth of fluency during the treatment period. Results varied between sets but provided evidence of increased fluency for most students.

Table 2

Changes in Student Fluency Scores on Divergent Thinking Exploration Questions (DTEQ)

Student	Change DTEQ 1 to 2	Change DTEQ 2 to 3
Student B	+21 points	−4 points
Student C	+2 points	−10 points
Student D	+3 points	−8 points
Student E	−7 points	+5 points
Student F	+43 points	+10 points

Student A was excluded from this analysis due to absences. Despite some decreases in scores between question sets, significant gains—such as Student F's total increase of 53 points—demonstrate that the treatments can increase mathematical fluency in the K-16 math classroom.

In conclusion, both the CMAT and DTEQ results provide preliminary evidence supporting the argument that purposeful integration of problem-posing and divergent thinking methods can effectively increase mathematical creativity in pre-service secondary mathematics teachers.

Conclusion

The project provides preliminary evidence that integrating these teaching methods and activities into the mathematics curriculum positively impacts creativity. However, the study had several limitations that future research should address. First, the small sample size and limited diversity, consisting of only five students, restrict the generalizability and reliability of the findings. Second, there were methodological concerns, including the absence of a control group and a lack of separation between treatment groups, which limited the ability to isolate the specific effects of the intervention; future studies should consider an ethically sound design where a control group receives the treatment after the post-test. Finally, attendance and scoring issues were present, as non-mandatory attendance may have impacted on consistent student engagement. Additionally, utilizing only the fluency section of Sheffield's rubric to monitor progress provided an incomplete picture of creativity development, which also includes flexibility and originality.

Future research must continue to integrate problem-posing and divergent thinking strategies into mathematics classrooms. Researchers should focus on developing better methods for measuring incremental improvements in all aspects of mathematical creativity throughout the treatment period, such as fully utilizing comprehensive rubrics like Sheffield's. By incorporating these methodologies into the K-16 curriculum, educators can cultivate a culture of exploration and innovation, strengthening essential problem-solving skills for future challenges.

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INVESTIGATING THE EFFECTS OF PROBLEM-POSING AND DIVERGENT-THINKING ON MATHEMATICAL CREATIVITY

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Problem-solving requires mathematical creativity, which is enhanced by divergent-thinking and problem-posing. This study explored which method is more effective at boosting creativity. Students in two Algebra I sections at an all-boys high school were split into two groups. One received six weekly divergent-thinking sessions, and the other received problem-posing sessions, both administered by the same professor. Pre- and post-evaluations were used to measure significant differences in mathematical creativity between the two groups.

Introduction

In a world that prizes problem-solving as an essential, employable skill, fostering mathematical creativity becomes crucial for addressing complex problems (Grégoire, 2016). The demand for problem-solving proficiency has led educators to seek techniques for strengthening students' abilities. Mathematical creativity, rooted in intellectual traits, can be significantly influenced by education, specifically by strengthening three key components: expertise, original thinking, and intrinsic motivation (Grégoire, 2016). The primary educational goal should be to encourage creative thinking, not just finding the 'right answer' (Mann, 2006).

The National Council of Teachers of Mathematics [NCTM] emphasizes this by making problem-solving the first of its five process standards, asserting it is both a goal of learning mathematics and an integral part of the practice (2000). Furthermore, this focus continued and was extended in 2010 with the Common Core Standards for Mathematical Practice (SMPs). Therefore, NCTM advises strengthening problem-solving via methods that improve mathematical creativity. The techniques of divergent-thinking and problem-posing are associated with enhancing creativity due to their varied and unique approaches to tackling problems (Mann, 2006). Exposing students to these creative methods is considered one of the most effective ways to enhance their mathematical creativity.

Literature Review

The concept of creativity remains challenging to define, with over a hundred contemporary definitions across fields like psychology and education (Mann, 2006; Treffinger et al., 2002). Definitions often emphasize either the process of being creative (Balka, 1974) or a tangible product (Haylock, 1997). Creativity has been described as a multifaceted construct involving divergent and convergent thinking, problem solving, and intrinsic motivation (Runco & Albert, 2010). Haylock (1997) categorized creativity into the mind (a special kind of thinking) or the hand (generation of products like art), though creative expression is also evident in science and innovation (Sriraman & Haavold, 2017). One widely referenced explanation comes from Torrance (1966), who defined creativity as a process of becoming sensitive to problems, deficiencies, forming hypotheses, testing them, and communicating the results. This definition later formed the basis for an operational definition of mathematical creativity focusing on constructing cause and effect hypotheses in mathematical situations (Mann, 2006).

Mathematical Creativity

Mathematical creativity is an interdisciplinary subset with its own varying definitions. It is often discussed in terms of problem formation, invention, independence, and originality, or measured by fluency, flexibility, and originality (Akgul & Kahveci, 2016; Silver, 1994). Sriraman (2009) synthesized it as a process that yields novel and/or insightful solution or the formulation of new questions that allow a problem to be viewed from a new angle.

Even though research mathematicians describe the field as highly creative and link it to the beauty of mathematics (Mann, 2006), many people wrongly view mathematics as solely about finding a 'correct answer.' Unfortunately, a sizable body of literature suggests students rarely experience mathematics creatively due to a lack of creative teaching approaches (Akgul & Kahveci, 2016; Mann, 2006). Creativity is difficult to strengthen when students are limited to rule-based applications and withheld from open exploration.

However, groups like the NCTM emphasize creativity as a crucial component of future education (Sriraman & Haavold, 2017). They envision classrooms that encourage students to approach problems from multiple perspectives and develop various representations to progress towards solutions (NCTM, 2000). To achieve this, students need open-ended experiences and opportunities to explore concepts beyond routine exercises (Mann, 2006). Two specific methods used to foster this necessary creativity are divergent-thinking and problem-posing.

Divergent-Thinking

Divergent thinking, originally proposed by Guilford and Torrance, is closely linked to creative thinking, with fluency, flexibility, originality, and elaboration as central features (Sriraman et al., 2017). It is defined as actively searching for diversity in problem-solving, often resulting from open-ended problems that allow for multiple solutions and original thoughts

(Kwon et al., 2006). Unlike convergent thinking, which seeks a single correct answer, divergent thinking welcomes numerous ideas, new strategies, and unexpected insights. For example, a study found that students asked to develop and answer problems using given concepts created and answered more questions with higher accuracy than those given traditional problems (Mann, 2006). While criticisms exist (e.g., convergent thinking can also be creative), most classroom questions still limit creativity by having only one answer. Open-ended problems, which are accessible to students of varied abilities, counteract this by allowing for divergent-thinking and cultivating more effective mathematical communication through discussion.

Problem-Posing

Problem-posing, the lesser-known counterpart of problem solving, is also acknowledged by many educators as essential for cultivating mathematical creativity (Fetterly, 2010; Sharma, 2014). It is the creation or reformation of problems and is seen as an essential means of mathematical education, forging mathematical curiosity. Problem-posing naturally aligns with inquiry-oriented instruction and requires open-ended problems (Fetterly, 2020; Silver, 1994). The process is highly beneficial, stimulating conceptual development, improving understanding of problem structure, and providing access to creative mathematics (English, 2003).

Problem-posing can follow a structured approach, such as Shaw and Aspinwall's (1999) five-step pose-and-probe rubric, which involves identifying a problem, listing its attributes, applying a 'what if not...?' question, investigating the new question(s), and relating the findings back to the original problem. This process demands the exploration of multiple methods and solutions, directly connecting to the three key measures of creativity: fluency, flexibility, and originality.

Evaluating Mathematical Creativity

Instruments used to measure mathematical creativity often involve open-ended problems or questions with open responses to allow for variability in the answer (Fetterly, 2010). These tests are typically scored based on at least three criteria: fluency, flexibility, and originality (Mann, 2006; Silver, 1994). Fluency is the quantity of relevant responses, regardless of quality. Flexibility is the number of distinct categories into which the ideas are organized. Originality is the ability to produce uncommon responses, often weighted more heavily for answers that appear less frequently in the sample population (Balka, 1974). Balka's Creative Mathematical Ability Test is a notable instrument that uses these criteria to analyze divergent responses, demonstrating the enduring importance of fluency, flexibility, and originality in assessing one's mathematical creativity abilities.

Methods

This study employed a pre- and post-test design to investigate the effects of two instructional treatments—divergent-thinking and problem-posing—on mathematical creativity in Algebra I students (Creswell & Creswell, 2018). The sample consisted of 44 male students from an all-boys high school, all taught by the same classroom teacher. One section of students received the divergent-thinking treatment, and the other received the problem-posing treatment. The same university professor delivered six 55-minute treatment sessions, one per week for six weeks, to both groups. Balka's Creative Mathematical Ability Test was used for both the pre- and post-evaluations to measure the effectiveness of the interventions.

Divergent-Thinking Treatment

The divergent-thinking group received six lessons emphasizing the importance of multiple methods and communicating mathematical concepts. The professor focused on open-ended problems, which allowed for multiple starting points, various paths, and occasionally multiple results. Core teaching techniques included: 1. Which One Doesn't Belong: Encouraging students to use mathematical vocabulary to justify which item does not fit with the rest; 2. Think-Pair-Share: A collaborative activity where students first reflect individually, then share ideas with a partner, and finally engage in a full-class discussion; and 3. Show and Tell: Requiring students to document and communicate their thought process, fostering deeper connections.

A major goal was to encourage students to find a solution and then, critically, to rethink through the problem from another approach. Students analyzed solved problems, determining if the methods used were novice or expert, and then generated further alternative perspectives. The core focus was on developing the maximum number of ways to think about a single problem to promote mathematical creativity (Kwon et al., 2006).

Problem-Posing Treatment

The problem-posing group's six lessons focused on the art of generating new problems rather than just solving given ones, a process closely linked to mathematical creativity (Silver, 1994). Lessons followed a pattern: individual work, Think-Pair-Share, and then a whole-class discussion. During discussions, the professor introduced the Pose-and-Probe rubric, based on Brown and Walter's (2005) work, leading students to identify an original problem's attributes and then ask 'what-if-not' questions to transform those attributes. Students explored these new, self-created problems, explicitly recognizing the process as problem-posing. Lessons reinforced three classroom procedures for problem-posing: challenge assumptions, change behaviors, and contaminate characteristics. The treatment aimed to develop the ability to generalize discovered mathematical patterns by transforming an initial problem into many others (English, 2003). These sessions also included short brain-breaks using inspiring quotes or fun facts to recharge and refocus the students' minds.

Findings

The study investigated the effect of divergent-thinking and problem-posing treatments on the mathematical creativity of 45 male sophomore students in an Algebra I course. The participants were predominantly White (95.6%), with a small percentage being Hispanic (4.4%). The restricted demographic profile suggests a need for future research with a more diverse sample.

Creativity Score Analysis

The study assessed mathematical creativity using scores in fluency (quantity of relevant responses), flexibility (number of response categories), and originality (uncommon responses). The difference between students' pre- and post-test scores was used to conduct a dependent t-test to determine whether there was a significant difference in the scores. The overall sample showed a statistically significant improvement from pre-test (M = 29.73, SD = 12.87) to post-test (M = 33.57, SD = 10.79), with $p < .05$.

Table 1

Descriptive Statistics for the Overall Sample

Test	*n*	*M*	Max	Min	Range	*SD*
Pre-Test	44	29.73	65	8	57	12.87
Post-Test	44	33.57	59	9	9	10.79

Treatment Group Outcomes

Further dependent t-tests were conducted for each class period's data. These tests showed that students in the problem-posing treatment (Table 2) improved their scores. When analyzed separately, the problem-posing group (n = 22) showed a significant improvement in scores from pre-test (M = 27.91, SD = 13.17) to post-test (M = 30.50, SD = 9.92), with a p-value of.027. This improvement was significant ($p < .05$).

Table 2

Descriptive Statistics for Problem-posing Treatment Group

Test	*n*	*M*	Max	Min	Range	*SD*
Pre-Test	22	27.91	65	8	57	13.17
Post-Test	22	30.50	59	17	42	9.92

Conversely, the divergent-thinking group (n = 22) also improved its average scores from pre-test (M = 31.55, SD = 12.59) to post-test (M = 34.14, SD = 11.81), but this difference was not statistically significant (p = .133; see Table 4).

Table 3

Descriptive Statistics for Divergent-thinking Treatment Group

Test	*n*	*M*	Max	Min	Range	*SD*
Pre-Test	22	31.55	64	12	52	12.59
Post-Test	22	34.14	59	9	50	11.81

The results indicate that the problem-posing treatment had a statistically significant effect on increasing mathematical creativity scores. However, it's worth noting that the non-significant divergent-thinking group ultimately achieved a higher average post-test score (34.14 vs. 30.50).

Table 4

Dependent t-Test Results

Pre - Post	*M*	*SD*	*SE*	*df*	*t*	*p*
Problem-Posing Treatment Group	-5.09	12.87	2.15	21	-2.37	.027
Divergent-Thinking Treatment Group	-2.59	13.17	1.66	21	-1.56	.133

In conclusion, we have evidence supporting the alternative hypothesis. Overall, after being exposed to treatment, students improved their scores significantly from pre- to post-test. The results also show that, when analyzed separately, the problem-posing treatment in the one group produced a significant difference in test scores, while the results from the other group (the divergent-thinking treatment group) did not produce a significant difference. However, it should be noted that, although the pre- and post-test scores of the divergent-thinking treatment group are not significantly different, their average post-test score is higher than the average post-test score of the problem-posing treatment group.

Conclusion

The development of mathematical creativity is a crucial educational goal, and the results of this study support the claim that specific instructional methods can enhance it. Both the problem-posing and divergent-thinking treatments resulted in score improvement, agreeing with literature that links these techniques to creativity development. Critically, the problem-posing treatment generated a statistically significant improvement in students' mathematical creativity scores, while the divergent-thinking group's improvement was not statistically significant.

Despite the positive findings, the study had limitations, primarily a small, exclusively male sample with a narrow demographic profile. Future research should use a larger, more diverse population to explore potential differences. While this study lacked a control group—due to ethical concerns about withholding beneficial treatments, including a group with normal instruction in a future design could provide stronger evidence for the significance of the observed gains. As mathematical creativity is key to problem-solving and appreciating the beauty of mathematics, treatments like problem-posing and divergent-thinking should be integrated into all mathematics classrooms to help students solve the unknown problems of tomorrow.

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UNDERSTANDING VS. TEACHING: HOW SECONDARY MATHEMATICS TEACHERS CHOOSE PROOFS

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This study examines how pre- and in-service secondary mathematics teachers in a research program evaluated proofs from both learner and teacher perspectives. The difficulties that secondary mathematics teachers experience with proof are well-documented. However, little research has examined what counts as proof from both a learner's and a teacher's perspective and what criteria can affect their decision. Findings show that teachers select proofs based on different criteria depending on whether the purpose is understanding or teaching. Reasoning, explanation, and justification were the primary criteria used for understanding, while accessibility and visualization were the primary criteria used for choosing proof for teaching.

Introduction

Given the central role secondary mathematics teachers play in implementing proof-related tasks, considerable attention has been devoted to ensuring that both pre- and in-service teachers learn proof in ways that prepare them to teach it effectively. Research has shown that teachers' knowledge of proof significantly influences students' opportunities to engage with proof-related tasks (e.g., Bieda, 2010; Stylianou et al., 2009). Secondary mathematics teachers' knowledge of teaching proof shapes both the opportunities they provide for students to engage in proof-related tasks and how they respond to students' proof productions (e.g., Bieda, 2010; Stylianou et al., 2009). Prior research on prospective and practicing teachers' proof evaluations has primarily examined their judgments about what constitutes a proof (e.g., Dickerson & Doerr, 2014; Knuth, 2002a, 2002b). However, these studies did not examine whether teachers evaluate arguments differently depending on whether they are reasoning as learners (for understanding) or as teachers (for teaching).

Because analyzing students' written work is essential for identifying strengths and weaknesses in mathematical reasoning (National Council of Teachers of Mathematics [NCTM], 2014), it is necessary to understand how pre- and in-service teachers evaluate arguments as proofs from a learner's perspective (for understanding) and a teacher's perspective (for teaching). Moreover, although teachers might not explicitly refer to different roles of proof (De Villiers, 1999), they might be sensitive to differences in contexts where there is a need for verification, for example, or one for communication. Teachers' criteria for choosing a proof over another can potentially shed some light into their perceptions of the role of proof. To address these issues, we examined how pre- and in-service secondary mathematics teachers, participating in an undergraduate research experience, evaluated arguments from the perspectives of understanding and teaching mathematical content. In this paper, we address the following research question: *What criteria do mathematics teachers use to evaluate proof for the purpose of understanding and teaching? How do the criteria used to evaluate proof for understanding differ from the criteria used to evaluate proof for teaching?*

Theoretical Perspective and Related Literature

Understanding how teachers choose and conceptualize proofs is crucial for effective mathematics instruction. Knuth's (2002a, 2002b) studies highlight the complex interplay between teachers' beliefs and pedagogical practices regarding proof in secondary school mathematics and were used as a framework for this study. Findings of these studies suggest that teachers' approaches to employing proofs are shaped by their own learning experiences and understanding of mathematical concepts.

Knuth (2002a) categorized the roles of proof in school mathematics as developing logical thinking skills, communicating mathematics, displaying reasoning, explaining why, and creating mathematical knowledge. All the teachers in this study agreed that formal methods of proof were most suitable for higher-level mathematics courses, whereas in lower-level courses, they viewed informal proofs as more appropriate. Many teachers in the study saw proof as an object of study rather than a tool for exploring and communicating mathematics, which may shape how they later teach proof to their own students. In Knuth (2002b), teachers' conceptions of the role of proof were categorized as: verification, explanation, communicating mathematical ideas, creating knowledge, and systematization. Among the roles mentioned by teachers, the promotion of understanding was notably absent. The results showed that teachers were often convinced by arguments lacking the structure of formal proofs. The features that teachers considered convincing included concreteness (i.e., using specific values), familiarity, level of detail, generality, showing why something works, and using a valid method.

Selecting proofs is a common teaching task, but the factors teachers consider in making these choices remain unclear (Sommerhoff et al., 2024). Dickerson and Doerr (2014) found that teachers differed in their views on the types of proofs suitable for classroom instruction: less experienced teachers preferred formal and rigorous proofs, whereas more experienced teachers favored concrete or visual proofs that are more accessible to students. A study conducted by Sommerhoff et al. (2024) found that teachers' selection of proofs for the classroom is influenced by multiple factors, including characteristics of the task and the proof itself, class characteristics, teacher characteristics, and the teaching and learning situation.

Teachers adaptively choose proofs that fit their students' needs, which may differ from the proofs they find most meaningful for their own understanding. Therefore, in this study, we explore the criteria teachers use when evaluating proofs for instructional purposes, and how these criteria may differ from those they apply when considering proofs for their own mathematical understanding.

Methodology

The 27 participants in our study, seven in-service teachers and 20 pre-service teachers, attended an 8-week NSF-funded Research Experiences for Undergraduates [REU]. The focus of the REU was to provide an authentic experience conducting research in mathematics, specifically an open problem in graph theory. The REU also included an education component. Participants were asked to complete a survey that included eight proofs of three mathematical statements. For the statement "complements of congruent angles are congruent," participants were given a paragraph proof [GEO-paragraph] and a two-column proof [GEO 2-column]. For the statement, "the sum of the first n positive integers is equal to n(n+1)/2," participants were given four proofs: a visual generalization based on forming triangular arrangement of squares and combining them to form a rectangle [SUM-visual], Gauss's well-known proof using equal addends [SUM-Gauss], a proof by induction [SUM-induction], and a generalization from a table of partial sums [SUM-table]. For the statement, "if $x > 0$, then $x + 1/x \geq 2$," participants were given a two-column algebraic proof of the converse [ALG-2-column] and a proof using a right triangle with legs of length $x - 1/x$ and 2 [ALG-geometric]. From these proofs, we asked participants to indicate which proofs (a) were most or least convincing, (b) were helpful to understand the mathematics involved, and (c) they would use in their classroom. For the purposes of this paper, we are comparing their answers to (b) and (c). Following Knuth's framework (2002a, 2002b) and building upon it, we analyzed the data to identify criteria the teachers used in selecting proofs for the purposes of understanding and teaching. We further analyzed the data to identify patterns in the criteria used by participants to select proofs for their own understanding or for teaching.

The survey responses were collected electronically, and the participants' written explanations provided qualitative data that helped identify their selection criteria. In the coding process, we first identified the criteria categories used by participants when selecting proofs for understanding and for teaching based on Knuth's framework (2002a, 2002b). We applied the constant comparative method (Glaser & Strauss, 1967) to refine our definitions and incorporate statements that did not fit Knuth's original criteria. Then, we examined patterns within and across these categories to organize the emerging themes.

Findings

Our results indicate that the participants used a variety of criteria when selecting proofs for understanding and teaching. Furthermore, the participants often used multiple criteria in selecting proofs, and their distribution varied based on the purpose of the proof. Table 1 provides the criteria that participants used in evaluating the eight proofs. Each category includes an example from the data.

Table 1

Categories of Critique

Category	Example
Proof Structure	"He provided his given information, the steps he used in more formal language, and the justifications for each step, and he wrote in a more formal form."
Accessibility	"It looks clean and is easy for students to follow. In addition, it is something they can all produce."
Generalization	"I would likely use the cases that show how the mathematician is thinking through base cases, finding simple patterns, and then how those patterns are applied for larger cases in general."
Visualization	"The visual approach ... shows [students] that the numbers actually represent something."
Connections	"Pictures that connect back to already familiar concepts."
Familiarity	"I found this to be one of the most convincing because I am the most used to it. This happens a lot in education because we get convinced simply because it is the norm."
Reason, Explain, Justify	"The proofs that utilize intuition and reasoning are more convincing to me ... I would rather hear someone's thought process over their ability to manipulate equations."

Visibility of Thinking	"I would use statement/reason proofs in my class simply because it helps break down the student's thought process during the proof to show what they are or are not understanding from the problem."
Multiple Strategies	"I feel like looking at all the proofs together helped me to understand the mathematics to the fullest because I was seeing multiple people explain why a problem makes sense"
Mathematical Language	"The proofs that used generalized language and symbology and made arguments that worked across all cases."

Once the critique categories were identified, we looked for patterns in their use in selecting proofs for understanding and teaching. These distributions can be found in Table 2. When selecting proofs that would help them understand, the teachers used a wide variety of criteria, with each criterion being used at least once. The most common criterion was reason, explain, justify, which was used 11 times in selecting six different proofs for understanding. For instance, one participant who selected the GEO 2-column proof stated, "proofs that gave statements and reasons why the statements were true" helped them to understand. While another participant who selected the SUM-Visual proof stated, "the proofs that best helped me understand the mathematics were less formal and instead focused on developing the patterns in the problem." This illustrates that although justification was deemed important, the proof that provided the best justification varied.

Table 2

Frequency of Criteria Used When Selecting Proofs for Understanding vs Teaching

	Understand	Teach
Proof Structure	8	4
Accessibility	7	9
Generalization	1	1
Visualization	8	9
Connections	2	4
Familiarity	2	3
Reason, Explain, Justify	11	1
Visibility of Thinking	1	3
Multiple Strategies	2	2
Mathematical Language	1	0

The categories of proof structure and visualization were each used eight times in selecting proofs that would aid understanding. Those who used proof structure as a criterion primarily chose a two-column proof, and those who used the visualization as a criterion exclusively chose the SUM-Visual proof. For instance, a participant who selected the GEO 2-column proof stated, “proofs that laid out the steps and the reasons for that step helped me break down the problems and understand the mathematics the most.” Conversely, a participant who selected the SUM-Visual proof stated the visual proof, “really helped me understand the mathematics because it was so visual. It gave a representation with blocks to showcase why this formula works.” The result illustrates a divide among the participants regarding what helped them with understanding—structure or visualization.

There were seven participants who mentioned accessibility as an important characteristic for understanding. However, the proofs the participants thought were accessible varied. For instance, a participant who selected the GEO 2-column proof stated it, “was helpful in understanding the mathematics because it was so clear and explicit.” Similarly, a participant who selected the SUM-Visual proof stated, “those that had a visual accompaniment helped me understand the mathematics because they were easier to follow.” Other proofs that the participants selected because they were accessible were the GEO-Paragraph proof and the SUM-Gauss proof.

For the purpose of understanding, the criterion of proof structure was primarily used to justify the selection of a two-column proof, and the criterion of visualization was only used in the selection of the SUM-Visual proof. In contrast, the criteria of reason/explain/justify and accessibility were used in selecting a variety of proofs. Participants differed on which proof they believed was most accessible or provided the best reasoning, explanation, and justification.

As with understanding, a variety of criteria, and often multiple criteria, were used in selecting a proof for the purpose of instruction. The most common criteria were accessibility and visualization, which were each used nine times in selecting a proof for instruction. Of the nine who used visualization as a criterion, eight selected the SUM-Visual proof as the one they would use during instruction. The one participant who did not select the SUM-Visual proof selected the GEO 2-column proof. They made the following statement, “I think that column proofs and pictures are easy to use in a high school classroom” that hinted at the need for visualization.

As we saw with the purpose of understanding, participants used the accessibility criterion to select a variety of proofs including the GEO 2-column, GEO-Paragraph, SUM-Visual, and

SUM-Gauss proofs. The participants seemed to differ in terms of which proof they thought would be easiest for students to comprehend. However, two participants specifically mentioned the need for multiple perspectives with an in-service teacher stating, "I try to make the proof accessible to my students. So, I try to use a variety because each student tends to gravitate to something different."

We saw fewer participants focus on structure ($n = 4$) for the purpose of instruction, as compared to understanding, but an increase in those who used the criteria of connections ($n = 4$), familiarity ($n = 3$), and visibility of thinking ($n = 3$). Of those that used connections as a criterion, two specifically talked about connections between the symbolic and graphical representations, and the other two talked about connections between different content ideas. The three participants that used the familiarity criterion did so in selecting the GEO 2-column proof. One participant stated that the two-column is the proof "I would use in my classroom since it was the two-column proofs that I remember from my Geometry class." Of the three that used the visibility of student thinking criterion, one selected multiple proofs, one the SUM Visual proof, and one the GEO 2-column proof. The participant who selected multiple proofs as making thinking visible stated, "the things they [the proofs] have in common is making the thinking REALLY visible."

Discussion

Perhaps the most salient finding in our study is that teachers use different criteria when choosing proofs for different purposes. The structure of the proof and criteria related to validity seemed more prevalent when choosing proofs from the personal perspective of understanding. However, emphasis on accessibility and visualization when choosing proofs for teaching may suggest that teachers are sensitive to different levels of formalism when thinking about the needs of their students. Reid and Knipping (2010) noted, "the level of students' understanding might be best improved by addressing teachers' understanding of proof itself, rather than exposing them to new methods of teaching about proofs and proving" (p. 71). Therefore, it is plausible that teachers' sensitivity to students' different needs is a result of participants' experience with researching open problems in mathematics as part of the REU. More specifically, teachers in our study seemed to consider an explanatory role of proof when choosing proofs for teaching.

Sommerhoff et al. (2024) have proposed a framework that identifies four major factors that possibly influence teachers' selection of proofs for use in teaching. In our study we are addressing one of these, *Task & Proof Characteristics*, but we must acknowledge that we are aware of the implicit influence of a second factor, *Teacher Characteristics* (e.g., beliefs, proof skills), which deserve a deeper examination. We hope our study contributes to the understanding of how teachers adaptively select proofs in their teaching.

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UNDERSTANDING PROOF: VIEWS FROM PRE-SERVICE SECONDARY MATHEMATICS TEACHERS AND INSTRUCTORS

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This paper examines how pre-service secondary mathematics teachers (PSMTs) and instructors of introduction-to-proof courses perceive proof. Among PSMTs, the most common roles of proof were verification and explanation, whereas instructors mostly saw proof as communication and verification. Both groups focused on modes of argumentation when describing proof characteristics. However, PSMTs emphasized different aspects of argument representation more than instructors did. These results suggest that introduction-to-proof courses should be intentionally designed to help PSMTs develop conceptions of proof that more closely match the practices of the mathematical community.

Introduction

Over the past decade, international research has demonstrated that mathematics teachers' ability to interpret and respond to students' proofs largely depends on their knowledge of how to teach proof (e.g., Lesseig et al., 2019; Sommerhoff et al., 2024). Studies consistently show that mathematics teachers across grade levels struggle with proofs (e.g., Ko & Rose, 2021), raising concerns about how mathematics teacher education programs prepare future teachers to help students understand proof, especially given policy documents that emphasize its importance (e.g., NGA & CCSSO, 2010). Scholars also emphasize the need for stronger collaboration between mathematicians and mathematics education programs. In the United States, ongoing shortages of secondary mathematics teachers mean many start teaching with limited preparation (Sutcher et al., 2016). As a result, undergraduates majoring or minoring in mathematics often begin teaching shortly after graduation. In this paper, pre-service secondary mathematics teachers (PSMTs) refer to mathematics teaching majors, middle school mathematics teaching majors, and mathematics majors or minors with enough background to teach at the secondary level. Introduction-to-proof courses that connect constructive mathematics (e.g., Calculus I, II, and III) and proof-intensive courses (e.g., Abstract Algebra, Linear Algebra, Number Theory) play a crucial role in preparing these future teachers. This study examines how PSMTs and instructors of introduction-to-proof courses conceptualize proof. Although previous research has explored PSMTs' conceptions of proof (e.g., Lesseig et al., 2019), little is known about how these two groups perceive proof. To address this gap, we investigate (Study 1) PSMTs' conceptions of mathematical proof and (Study 2) instructors' views of proof within mathematics, comparing the two to identify key similarities and differences in their understanding.

Theoretical Perspective

Proof is often described as a mathematical object—a finished argument that establishes the truth of a claim (Balacheff, 2008). While verification confirms correctness, it offers limited insight into underlying mathematical ideas (e.g., Knuth, 2002a, 2002b). Accordingly, mathematicians emphasize proof's explanatory role—its capacity to reveal why a statement is true (e.g., De Villiers, 1990). From this perspective, proof functions not only as a product but also a communicative action aimed at explaining mathematical reasoning and establishing shared mathematical understanding (Knuth, 2002b). Despite these multiple roles, proof is characterized by accepted statements, valid reasoning, and clear logical representations (Stylianides, 2007). Accepted statements include definitions, axioms, and theorems that require no additional proof (e.g., Stylianides, 2007). Also, modes of argumentation include logical deduction, consideration of all cases, and techniques such as proof by contrapositive or by contradiction (e.g., Stylianides, 2007). Individuals at different stages of participation in the mathematics communities may emphasize these aspects differently. Pre-service secondary mathematics teachers (PSMTs) may conceptualize proof as a static object defined by structures and correctness. However, instructors may implicitly view proof as the communicative norms of the mathematics community by audiences' expectations and shared standards of rigor. This study builds on previous work regarding the roles of proof (e.g., De Villiers, 1990; Knuth, 2002a) and components of proof (e.g., Lesseig, 2016; Steele & Rogers, 2012), summarized in Figures 1 and 2.

Methods

Study 1: PSMTs' Perceptions of Proof

Participants were recruited from two sections of the transition-to-proof course Discrete Mathematics (Fall 2020 and Fall 2021). In Fall 2020, 11 PSMTs agreed to participate, including three middle school mathematics teaching majors, two mathematics teaching majors, two mathematics majors, and four mathematics minors. In Fall 2021, 20 PSMTs participated, comprising three middle school mathematics teaching majors, seven mathematics teaching majors, six mathematics majors, and four mathematics minors. Only two students from Fall 2020 declined to participate. Participants completed a written class activity titled "Getting to Know You" on the first day, which asked: "What is a mathematical proof?" Their written responses served as the primary data source. Two of the authors independently reviewed all 31 responses to identify perceived roles and characteristics of proof, guided by an established analytical framework. Coding reliability was maintained through comparison and discussion until full agreement was reached.

Figure 1

Roles of Proof (Adapted from De Villiers, 1990)

Role/Function of Proof	Coding Definition	Language Used to Describe this Role/Function in de Villiers (1990)
Verification	Proof as a means to obtain conviction and establish the truth of a mathematical statement	• Verification of the correctness of mathematical statements (p. 17) • Conviction or justification (p. 17) • The idea is that proof is used mainly to remove either personal doubt and/or those of skeptics (p. 17) • Making sure (p. 17) • Concerned with the truth of a statement (p. 17)
Explanation	Proof as a means to promote understanding and illumination of why underlying mathematical concepts are true	• Why it may be true (p. 19) • Psychological satisfactory sense of illumination, i.e. an insight or understanding into how it is the consequence of other familiar results (p. 19) • Gives an understanding (p. 20) • One which makes us wiser (p. 20) • Convey an insight into why the proposition is true (p. 20)
Systematization	Proof as a means to structure unrelated definitions and previously-proved results to gain a global perspective of mathematical concepts	• The inclusion of a result in a deductive system (p. 19) • Systematization of various known results into a deductive system of axioms, definitions and theorems. (p. 20) • Intricately involved in the mathematical processes of a posteriori axiomatization and defining, … which form the backbones of both local and global systematization (p. 20) • Organize logically unrelated individual statements which are already known to be true, into "a coherent unified whole" (p. 21) • The focus falls of global rather than local illumination (p. 21)
Discovery	Proof as a means to expose unexpected results beyond the given mathematical scope or context	• New results are discovered/invented in a purely deductive manner (p. 21) • Proof can frequently lead to new results (p. 21) • To the working mathematician proof is therefore not merely a means of a posteriori verification, but often also a means of exploration, analysis, discover and invention (p. 21) • Generalize the result [to a broader class] (p. 21) • Deductive discovery via deductive generalization (p. 22)
Communication	Proof as a means to transmit mathematical thoughts and strategies to others	• A form of discourse, a means of communication among people doing mathematics (p. 22) • A human interchange based on shared meanings (p. 22) • Creates a forum for critical debate (p. 22) • The social process of reporting and disseminating mathematical knowledge in society (p. 22) • Subjective negotiation of not only the meanings of concepts concerned, but implicitly also of the criteria for an acceptable argument (p. 22)

Study 2: Introduction to Proof Instructors' Perceptions

To explore how instructors of introductory proof courses define, evaluate, and describe proof in mathematics, a survey was developed and refined based on feedback from two experienced instructors. The final survey consisted of three sections: Demographic Information, covering doctoral background, institution type, teaching experience, and courses taught; Proof Evaluation, where participants defined proof, assessed five student arguments, and ranked those arguments;

and Roles of Proof, in which they ranked De Villiers' (1990) five roles of proof (verification, explanation, systematization, discovery, and communication) and described how they incorporate them into teaching. During Spring 2024, emails were sent through the Service, Teaching, and Research (STaR) listserv and the Research in Undergraduate Mathematics Education (RUME) community. The email described the study and outlined eligibility criteria. A total of 45 valid responses were collected over two and a half months. Among these participants, 27 held doctorates in mathematics, 12 in mathematics education, and six in related fields (e.g., STEM education, applied mathematics). The focus of analysis in this paper was on participants' written responses to "In your own words, what is a mathematical proof?" The other two authors independently coded responses using the same analytical framework as in Study 1. They then discussed discrepancies to reach consensus on the final categorization. This qualitative approach was appropriate for capturing participants' personal conceptions of mathematical proof and for allowing direct comparison across pre-service teachers and instructors using a shared analytical framework.

Figure 2

Characteristics of Proof (Adapted from Stylianides, 2007)

Characteristic of Proof	Language Used to Describe this Characteristic in Stylianides (2007)	Coding Example Used to Describe this Characteristic in Stylianides (2007)	Coding Characteristic
Set of Accepted Statements	[Proof] uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification (p. 291).	Definitions, axioms, theorems, etc. (p. 292)	Accepted Truths
Modes of Argumentation	[Proof] employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community (p. 291).	Application of logical rules of inference (such as modus ponens and modus tollens), use of definitions to derive general statements, systematic enumeration of all cases to which a statement is reduced (given that their number is finite), construction of counterexamples, development of a reasoning that shows that acceptance of a statement leads to a contradiction, etc.	Valid Methods Logical Structures Sufficient Details Generality Completion
Modes of Argument Representation	[Proof] is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community (p. 291).	Linguistic (e.g., oral language), physical, diagrammatic/pictorial, tabular, symbolic/algebraic, etc.	Organization/Format Mathematical Terminologies

Table 1

Comparisons between PSMTs' and Introduction-to-Proof Instructors' Roles of Proof in Mathematics

Role of Proof	Frequency of PSMTs	Frequency of instructors
Verification (V)	20 (38%)	31 (30%)
Explanation (E)	20 (38%)	24 (23%)
Communication (C)	6 (11%)	35 (34%)
Systemization (S)	7 (13%)	11 (11%)
Discovery (D)	0 (0%)	2 (2%)
Totals[1]	53 (30 PSMTs)	103 (44 Instructors)

[1]Totals may include multiple counts for a single PSMT or instructor (i.e., a PSMT's response might have been placed under more than one role of proof).

Results and Discussion

Tables 1 and 2 summarize how PSMTs and introduction-to-proof instructors described the roles and characteristics of mathematical proof. The percentages in the second and third column represent the proportion of respective groups that included that particular role of proof when responding to the survey question. Almost all instructors (44 of 45) referenced at least one of De Villiers' (1990) roles of proof, while no PSMTs and only 2% of instructors mentioned the discovery role. This limited focus on discovery likely reflects common instructional practice, where proof is rarely used to generate new insights unless exploratory tasks are intentionally included in the process.

Roles of Proof

According to Table 1, 30 of 31 PSMTs provided responses reflecting one or more roles such as verification, explanation, systematization, and communication. Verification and explanation were the most common, each accounting for 20 out of 53 coded cases. Most PSMTs viewed proof mainly as a tool for verification. For example, Jenny described proof as "a definitive argument declaring a mathematical statement is true," and Addison wrote that proof "uses theorems and other math constants to describe a problem as true." These findings align with previous research (Knuth, 2002a; Lesseig et al., 2019), which shows that many PSMTs see verification as the main purpose of proof, likely because of the traditional focus on confirming correctness in school mathematics. However, mathematicians emphasize that proof should not only verify statements but also foster understanding and insight (e.g., Bleiler-Baxter & Pair, 2017; Knuth, 2002a). Among instructors, communication emerged as the most frequently identified role (34% of 103 cases), followed by verification (30%). For example, Alex stated, “A proof is a formal discourse,” while Brooke described it as “a justification of a mathematical claim that conforms to mathematical communication norms.” Similarly, Justin explained that proof is “a tool to communicate to yourself and others … that a mathematical statement is true.” This difference, while not statistically tested, is educationally meaningful and highlights a divergence between PSMTs’ initial views of proof and instructors’ conceptions of proof as a communicative practice within the mathematical community (e.g., De Villiers, 1990; Knuth, 2002a). Both groups rarely referenced the systemization role (13% of PSMTs; 11% of instructors), suggesting limited recognition of proof as a tool for organizing mathematical knowledge across topics.

Table 2

Comparisons between PSMTs’ and Introduction-to-Proof Instructors’ Characteristics of Proof in Mathematics

Characteristics of Proof	Frequency of PSMTs	Frequency of instructors
Set of Accepted Statements	10 (26%)	18 (31%)
Modes of Argumentation	15 (38%)	30 (52%)
Modes of Argument Representation	14 (35%)	10 (17%)
Total[1]	39 (23 PSMTs)	58 (43 Instructors)

[1]Totals may include multiple counts for a single PSMT or instructor (i.e., a PSMT’s response might have been placed under more than one characteristic of proof). Percentages reflect the proportion of participants in each group associated with a given characteristic of proof, while frequencies report how many participants in each group were associated with a given characteristic of proof, independent of how many times it occurred within an individual response.

Characteristics of Proof

According to Table 2, 23 of 31 PSMTs and 43 of 45 instructors referred to at least one of Stylianides’ (2007) characteristics of proof: modes of argumentation, modes of argument representation, and a set of accepted statements. For PSMTs, modes of argumentation was the most frequently identified characteristic (15 of 39 coded cases). Participants often emphasized logical reasoning and structured progression. For example, Isabella described proof as involving “deep logical thought,” Emmalee defined it as “a series of steps that can be verified for accuracy and rules of logic,” and Sofia wrote that it requires “assessing what you know and showing the steps of your thought process.” This focus on procedure and structure may reflect reliance on memorization and formal reasoning processes emphasized in proof-intensive coursework (e.g., Harel & Sowder, 2007) and classroom expectations to show all work (Dickerson & Doerr, 2014). Among instructors, modes of argumentation were also the most frequently cited characteristic (30 of 54 coded cases). Elizabeth explained that proof “should be sufficiently rigorous to make it clear … that there are no logical gaps,” while Kelsey and Matt described it as a “logical and well-written argument”. These responses highlight instructors’ focus on clarity, logical structure, and coherence in mathematical reasoning. Across both groups, PSMTs mentioned modes of argument representation more often than instructors (35% vs. 17%), reflecting attention to how proofs are written and presented. Both groups referenced the use of accepted statements at similar rates (26% of PSMTs; 31% of instructors), showing a shared understanding that proofs rely on established axioms, definitions, and theorems.

Conclusion

Overall, our findings suggest that PSMTs enter transition-to-proof courses with conceptions of proof centered on verification, explanation, modes of argument, and representations of argument. However, instructors tend to hold broader views that include communication, verification, explanation, modes of argument, and a set of accepted statements. Understanding PSMTs’ initial perceptions can help instructors of introductory proof courses design learning experiences that intentionally expose PSMTs to a wider variety of proof roles and characteristics. This approach enables transition-to-proof courses to better support PSMTs in forming conceptions of proof aligned with mathematical practice and in preparing them for future roles as mathematics teachers.

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