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RCML History

The Research Council on Mathematics Learning, formerly The Research Council for Diagnostic and Prescriptive Mathematics, grew from a seed planted at a 1974 national conference held at Kent State University. A need for an informational sharing structure in diagnostic, prescriptive, and remedial mathematics was identified by James W. Heddens. A group of invited professional educators convened to explore, discuss, and exchange ideas especially in regard to pupils having difficulty in learning mathematics. It was noted that there was considerable fragmentation and repetition of effort in research on learning deficiencies at all levels of student mathematical development. The discussions centered on how individuals could pool their talents, resources, and research efforts to help develop a body of knowledge. The intent was for teams of researchers to work together in collaborative research focused on solving student difficulties encountered in learning mathematics.

Specific areas identified were:

1. Synthesize innovative approaches.
2. Create insightful diagnostic instruments.
3. Create diagnostic techniques.
4. Develop new and interesting materials.
5. Examine research reporting strategies.

As a professional organization, the **Research Council on Mathematics Learning (RCML)** may be thought of as a vehicle to be used by its membership to accomplish specific goals. There is an opportunity for everyone to actively participate in **RCML**. Indeed, such participation is mandatory if **RCML** is to continue to provide a forum for exploration, examination, and professional growth for mathematics educators at all levels.

The Founding Members of the Council are those individuals that presented papers at one of the first three National Remedial Mathematics Conferences held at Kent State University in 1974, 1975, and 1976.

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Framing Equitable Learning

FACTORING QUADRATICS: HOW TRACKING SHAPES TEACHERS' INSTRUCTIONAL DECISIONS AND VIEWS OF STUDENTS

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NCTM strongly advocates to end tracking, highlighting the various detrimental effects such a structure has on students and teachers alike (NCTM, 2018). To understand how tracking influences teachers' instruction of a particular topic, we interviewed nine experienced secondary teachers about how they teach factoring of quadratics. Results indicate that teachers invest a significant amount of work learning different black box procedures which they present to lower-tracked students but not to honors classes. We provide a framework highlighting three different ways tracking shapes teachers' decision making and their views of students more generally.

Tracking has become the norm in the US with recent NAEP data showing that 75% of 8th students are tracked in mathematics (Loveless, 2012). So ubiquitous is this practice, that it is almost taken for granted that the best way to support students with perceived differences in ability is to place them in homogeneous environments, designed for their particular aptitude (Ansalone, 2009). However, there is limited evidence that such homogeneous grouping leads to more academic success (Boaler, 2013). In fact, while there are few examples of schools that have moved away from tracking, collective results indicate that heterogenous grouping leads to improved performance for lower-achieving students without negatively affecting higher achieving students (Rui, 2009). Consequently, many educators now advocate for the dismantling of such a structure, highlighting the various detrimental effects such labels have on students as well as the outsized role they have on teachers' perceptions (NCTM, 2018). Nonetheless, while a few studies have documented the different learning environments between different tracks (e.g., Boaler et al., 2000), we do not have a detailed characterization of how tracking shapes teachers' orientation or their instructional decisions around a particular topic. The goal of this study was to fill this gap and answer the following research question: *How does tracking affect how secondary teachers plan instruction and view their students?*

Factoring

To explore this question, we chose the content area of quadratic factoring. Factoring is at the core of the algebraic understanding outlined by the Common Core State Standards (2010) as the ability to "produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression" (p. 36). However, learning to meaningfully factor an expression can be challenging. In most cases, the process cannot be determined directly, but requires reversing the distributive property, observing patterns that result in specific forms, and then undoing this process, a method often referred to as guess and check. For quadratics in the form $ax^2 + bx + c$, while the underlying patterns are quite straightforward when $a = 1$, this is not the case when $a \neq 1$. Consequently, several alternative algorithms have emerged that consist of various unexplained, black box steps (e.g., Slide and Divide Method). These methods not only foster a view of algebraic expressions as disconnected strings of objects, but fail to develop a robust view of procedural fluency. Such a range of approaches provided a fruitful context to analyze teachers' instructional views and decision making.

Literature

Tracking students based on mathematical achievement is quite prevalent in the US. However, many scholars now question the effectiveness and fairness of this practice. While viewed as an appropriate, if not necessary, accommodation to meet the varying needs of students, it leads to the distribution of students in ways that correlate strongly with inequities based on race, gender, and socioeconomic status found in our broader society (Augyus et al., 1996; NCTM, 2018). Such patterns call into question the effectiveness of accurately identifying aptitude, as achievement is influenced by other nonacademic factors, often reflecting opportunity gaps or other systemic biases (Stiff & Johnson, 2011). Moreover, placement is regularly based on notions of achievement that are grounded in a narrow, fixed view of mathematics ability, which contradicts the rich, multi-dimensional view that most math educators promote. Furthermore, teachers' own biases can come into play, as student evaluations have been found to elevate the placement of white students relative to their achievement, but not black students (Faulkner et. al., 2014).

The inequities are further exacerbated due to the instructional differences between higher and lower tracks, thus expanding the exact differences tracking aims to address. For students in lower tracks, instruction often consists of rote algorithms (NCTM, 2018), characterized by an emphasis on obedience and discipline. In contrast, teachers of higher tracks are more likely to guide their instruction in a way that "cultivates their mathematical identities, conceptual understanding, and critical problem solving and thinking skills" (NCTM, 2018, p. 17). Contributing to such issues, teachers themselves are tracked, with less experienced and less qualified teachers often allocated to lower-level classrooms (Augyus et al., 1996; Nirode & Boyd, 2023).

In addition, tracking sends problematic messages of a fixed mindset, leading students in both tracks to question their sense of belonging. Not surprising, Francis et al. (2020) found a strong relationship between track labels and self-confidence in mathematics, even when controlling for prior achievement. However, studies have highlighted that detracking can foster lower self-esteem among lower achieving students, thus suggesting that instructional practices that encourage individual accomplishments rather than competition are essential in the process of detracking (Chmielewski et al., 2013). For students placed in higher tracks, the results have also been mixed (Riu, 2009). Yet, students, particularly girls, in higher tracks often see challenge as a threat to their ability, viewing struggle as evidence that they do not belong, consequently avoiding challenges, not asking for help, and hiding their mistakes (Dweck, 2008).

While tracking has adverse effects on students, it also plays an outsize role in teachers' perceptions and instruction. While we do not have detailed descriptions of how tracking shapes teachers' instructional decisions, we know that many teachers base their expectations of student performance on their assigned track, even when presented evidence to the contrary (Ansalone, 2009). Not surprising, they have higher expectations of students placed in higher tracks and convey these expectations to students verbally in their instruction (Ansalone, 2009). On the contrary, teachers report avoiding giving challenging problems to their students, in general, but more specifically to lower levels, due to a fear of student reaction, and a lack of self-efficacy in ability to support students (Darragh, 2013). Consequently, they fail to encourage struggle in their classroom and instead focus on speed and accuracy (Darragh, 2013).

Methods

To explore how tracking shapes teachers' instruction of quadratic factoring, we conducted and videotaped one-hour individual semi-structured interviews (Ginsburg, 1997) with nine high school teachers (1 male, 8 female; all white). Overall, the teachers possessed a wealth of experience and expertise. Except for one teacher who had just completed her first year of teaching, the other teachers had four to 34 years in the classroom with an average of 17 years teaching. All but one had a graduate degree, with the majority studying specifically mathematics education, and had experience teaching all tracks offered at their schools across multiple subjects. Interview questions focused on teachers' familiarity and understanding of different factoring methods, their use of different tools (i.e., algebra tiles) and representations and ways they differentiated instruction. To ensure a diversity of perspectives, teachers were selected from seven different schools across two different states. The schools represented a wide range of student populations in terms of their socioeconomics (30%-88% poverty rates) as well as prior achievement (27%-65% meeting minimal proficiency levels on state algebra test). In particular, two schools were among the top performing in the state and two were among the lowest in the state. However, the one consistent factor across the schools was the use of tracking, with all using at least two levels and some three levels for each course.

Data Analysis

To understand different elements that affected the teachers' instruction of factoring, we began our analysis by identifying all statements that referenced their rationale, orientation, or understanding and categorizing these comments into different areas. Although none of our questions asked specifically about instructional differences based on levels, the teachers' unsolicited responses about tracking made it clear that such a structure played an outsized role in their decision making. We then went back through the interview data, transcribing and analyzing all moments when teachers spoke specifically about tracking. Using a constant comparison method of looking across subjects, we identified trends and differences in the methods they taught, their rationale for why, and their characterization of students in different levels (Strauss & Corbin, 1994). This gave rise to an initial set of codes. We refined these codes through multiple iterations of analysis and discussion, ultimately identifying different categories for how teachers used tracking to guide their instruction and how they viewed students in these different tracks.

Results

We organized the results around the two significant findings that emerged from the data. The first was simply the importance that the familiarity with a wide range of procedural shortcuts played in the teachers' instruction of lower-level courses. Of the nine teachers, all but one, a first-year teacher, knew multiple, alternative black box algorithms for factoring quadratics. However, while each teacher could correctly carry out the different algorithms, not one could provide a rationale for various steps involved. Nonetheless, they all used their knowledge of these algorithms to differentiate instruction, providing certain students mathematically limited methods while engaging others in more meaningful guess and check approaches. Notably, all but two teachers based such instructional decisions on the track assigned to students. This discrepancy means that rich mathematical thinking was almost exclusively reserved for those students placed in honors classes. Moreover, although these teachers worked in schools consisting of students with vastly different prior achievement levels, this pattern of teaching starkly different methods to the two different levels was universal, emphasizing that the label assigned students drove teachers' instructional choices much more than their students' understanding. Furthermore, such consistency across the different teachers and schools, indicates the value black box algorithms had for these secondary teachers. Of the range of instructional knowledge and tools available to support students in factoring, all of them had devoted time and energy learning these procedures.

In addition to the strong emphasis on non-mathematical procedures in lower-tracked courses, we also identified three different categories, *responsive*, *predetermined instruction*, and *deficit*, that characterize the different ways tracking informed teachers' instructional decisions as well as their views of students. The first group, responsive, consisted of teachers who distinguished themselves by their consistent attitude toward teaching across tracks. While these teachers' instruction was not responsive in a way that aligns with literature (Robertson et al., 2015), we use this term aspirationally as they did distinguish themselves by presenting different factoring methods to all of their students, regardless of track, in an attempt to find one that students could use successfully. For example, one teacher in this category, to illustrate her rationale for picking a particular method, noted that "if the students are really good at GCF then the grouping method is a great method because… they are good at the GCF part". Similarly, the other teacher in this category highlighted that she usually teaches multiple methods to all of her classes until they find one they like. Again, this teacher did not base her instruction on predetermined levels, but was aware of the understanding her students possessed and chose the method(s) that she felt would best meet their needs. In addition, we found that these two teachers did not actively avoid struggle but rather acknowledged its role in learning. Although algorithms that circumvent struggle were present in their instruction, they did not teach these methods for that purpose. Notably, although only two teachers fell into this category, they worked in two, very different schools in terms of socioeconomic status and performance of their students, indicating that such decisions were a result of their orientation. In addition, these two teachers differed considerably in their views of conceptual understanding. While the first teacher stressed the importance of "avoiding the tricks" for students who have lower proficiency, the other teacher used more procedural methods to respond to students' struggles. Nonetheless, their approach to tracking was consistent in that they relied on their students' performance to determine their instruction, not the assigned groups.

The second category, which we denoted as predetermined instruction, consisted of teachers who consistently reported presenting conceptually distinct factoring methods to students in different levels. Throughout the interview, these teachers repeatedly described their instructional approach to factoring as distinct for upper and lower tracks, believing such differentiation would better meet the needs of what they saw as two different populations of students with two different mathematical trajectories. Often, they would explain the difference in their instructional approach quite pragmatically, highlighting how the course demands moving forward differed for the various tracks. For example, one teacher said that "every level is going to be completely different, you're in my calculus class you better not be pulling out grouping, you better not be pulling out that box, you don't have time for the box." Similarly, these teachers defended their choice to use more procedural methods based on the need for continuity, noting, for example that the box method aligned with the previous way lower tracked students had been taught to multiply binomials. In addition to believing different factoring methods were appropriate for the two groups, these teachers also differentiated their instruction between tracks by the number of methods they would present. As one teacher described, "I try to stick with one method in [lower level], and in honors classes I will show them multiple methods and kind of let them decide what they prefer." Such discrepancies further illuminate the access to different mathematical thinking they provided students in different tracks.

Finally, the last category, which we refer to as deficit, was characterized by teachers who not only implemented different methods, but ascribed character traits to students based on their track to explain their instructional decisions. For example, teachers described their higher tracked students as "able to intuit things very easily," "smart enough to understand [even when an explanation is not given]," and "eat [factoring] up like candy." On the contrary, they characterized their lower tracked students as wanting a more "concrete method," "not liking to be wrong on their first try," and "not being wired… to try and play with numbers." Their instructional rationale seemed less pragmatic, and more based on a belief that these students were simply unable to make sense of conceptual mathematics or unwilling to persevere. When asked why they do not teach guess and check to lower levels, one teacher replied that "it just wastes time for 99% of these students to explain that there is sound math behind it." Another, said that such students couldn't do guess and check, "they will only come up with one option…they just get frustrated and tend to quit." As this latter comment illustrates, these teachers chose to teach algorithmic methods to lower levels purposefully to avoid struggle. As one teacher explained, algorithms are "clean and simple for them…it is less work, it is shorter, it is more straight forward." It was almost as if they believed their role for teaching non-honors classes was to remove the cognitive demand for students and minimize any challenges. Finally, these teachers seemed to look for evidence to affirm characteristics they attributed to different labels. At one point, a student who the teacher believed did not belong in honors inadvertently interrupted the interview. The teacher noted that this student had struggled with trial and error and used her difficulty as evidence that she was not an honors student and that such a method is "best for the top kids."

Conclusion

In a letter to the NCTM membership, Berry (2018) noted that a successful end to tracking would require a range of changes including a) shifts in teachers' beliefs about who is capable of doing and understanding mathematics and b) equitable instructional practices. Our analysis of how tracking shapes teachers' instruction of factoring confirms this need illustrating how this structure plays an outsized role in teachers' instruction and orientation, regardless of their school or their students. While some teachers listened and responded to the needs of their students, most used students' assigned track to dictate both their instructional decisions and views of students. Moreover, to meet the perceived needs of students who have been labeled as less mathematically proficient, teachers have collectively produced and distributed a range of algorithms that serve to only limit the access to mathematical reasoning these students receive. Such findings highlight the need to help teachers develop instructional methods that go beyond teaching factoring as a system of rules and instead engage students in exploring patterns to engender different mathematical practices such as *making use of structure* and *persevering in problem solving* (CCSS, 2010). Rather than spending energy creating shortcuts that limit mathematical reasoning, teachers need guidance in how to use instructional tools like algebra tiles, which support students in productive struggle and explore connections without removing the thinking. In addition, teachers need help reflecting on how tracking shapes their instruction, learning to attend to students' thinking not the level they have been placed. We believe the framework that emerged in this work can be a useful tool in this work.

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LEARNING STRATEGIES FOR MATH GROWTH MINDSET, SELF-REGULATION AND PERFORMANCE

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This study examines the effectiveness of integrating learning-strategy instruction within the content of gate-keeper math courses in fostering a math growth mindset and self-regulated learning (SRL) in underrepresented minority students and its implications on students' performance. We propose and explore innovative ways to seamlessly integrate evidence-based cognitive, metacognitive, and management learning strategies within the course via the presentation of course material, class discussions, and assignments. Our conceptual framework provides a model for understanding the interrelationships between four constructs: learning strategies, math mindset, SRL, and performance, while accounting for the students' racial, gender, and math identities.

Introductory mathematics courses are the cornerstone courses for STEM disciplines. They provide students with the quantitative training needed for their STEM majors, all of which are becoming more quantitative in response to a rapidly changing data-driven job market. Students' performance in these foundational math courses profoundly impacts their transitions from high school to college, their ability to remain enrolled, make progress, and ultimately graduate (Carver et al., 2017). Nonetheless, enhancing students' learning experiences and achievements in these critical gateway courses has posed a persistent challenge for higher education institutions throughout the country. This challenge is even more pronounced for minority-serving institutions, including Historically Black Colleges and Universities (HBCUs).

Extensive research has explored the influence of math growth mindset and self-regulated learning (SRL) on the academic performance and persistence of STEM students (e.g., Yeager et al., 2019). Similarly, a wealth of literature highlights the positive correlations between learning strategies and SRL, as well as the link between instruction in learning strategies and enhanced academic performance (Donker et al., 2014; Weinstein et al., 2000). The majority of these investigations have centered on K-12 students. As a result, there is a notable gap in knowledge concerning the effectiveness of innovative approaches to integrate learning-strategy instruction within math courses, with the aim of fostering a math growth mindset and SRL among HBCU students. Furthermore, it remains unclear whether math growth mindset, SRL, or a combination of both, function as mediating factors in the relationship between learning-strategy instruction and the performance of HBCU students in math courses.

The present study explores innovative ways to seamlessly integrate evidence-based cognitive, metacognitive, and management learning strategies within four gate-keeper math courses (College Algebra I/II and Calculus I/II) via the presentation of course material, class discussions, and assignments. Data is collected to evaluate the efficacy of learning-strategy instruction in fostering a math growth mindset and SRL in underrepresented minority students and its implications on students' performance in these courses. The research is well-timed, as the demand for knowledge and application of learning strategies, fostering a growth mindset, and embracing SRL is paramount for academic success in the post-COVID era.

Related Literature and Framework

The belief that math ability is inherent has been ingrained in the U.S. (Stevenson et al., 1993). This belief constitutes a fixed mindset and contributes to the persistent problem of underachievement and low participation in math (Boaler, 2013), especially among females, African Americans, and Latinx students (Flores, 2007; Sun, 2015). On the contrary, growth mindset is the belief that intelligence can be increased through one's efforts (Dweck, 2000, p.3).

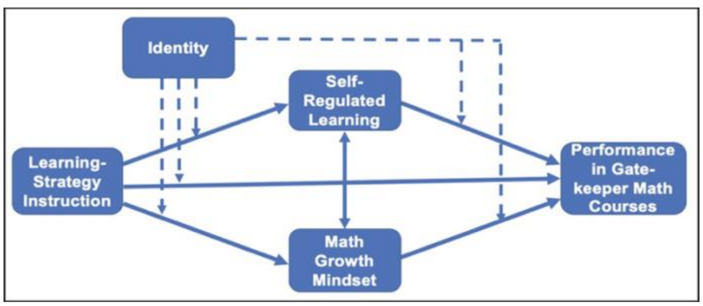
Another factor influencing students' math performance is whether they employ SRL (e.g., Fauzi & Widjajanti, 2018). Self-regulated learners are characterized by their ability to be metacognitively, motivationally, and behaviorally active participants in their own learning process (Zimmerman, 2000). They are actively involved in maximizing their opportunities and abilities to learn and can critically evaluate and intentionally alter how their thoughts, behaviors, and working environments contribute to their learning outcomes (Darr & Fisher, 2015). Self-regulation is also crucial for mathematical problem-solving (Marchis, 2012).

According to literature, students' knowledge and use of learning strategies can be a common facilitator of both constructs of math growth mindset and SRL, which would in turn lead to improvements in students' performance in math. Learning strategies, as defined by Pressley et al. (1989), refer to "processes (or sequences of processes) that, when aligned with the requirements of tasks, enhance performance." These strategies encompass students' thoughts, behaviors, or beliefs that facilitate the acquisition, comprehension, or practical application of new knowledge and skills (Weinstein et al., 2000).

Our proposed conceptual framework is poised to serve as a model for future research examining the intricate connections among instructional strategies for learning, the development of a math growth mindset, SRL, and math performance. This framework postulates the following hypotheses: (i) learning-strategy instruction has a direct effect on students' performance (Donker et al., 2014); (ii) learning-strategy instruction can indirectly influence performance by fostering students' math growth mindset, leading them to perceive new avenues for growth in learning and achievement, and students' SRL (McDaniel & Einstein, 2020); (iii) a bidirectional association exists between math growth mindset and SRL; and (iv) students' various social identities (such as racial, gender, and math identities) are likely to moderate the relationships described in (i)-(iii). The bidirectional relationship in (iii) has not been tested in the literature and is motivated by the results of Burnette et al. (2014) who concluded that the associations between mindsets and self-regulation are not straightforward.

Fig 1

Conceptual Framework.



Guided by the above framework, the study will address the following research questions.

Table 1

Research Questions.

RQ1	To what extent does learning-strategy instruction in gate-keeper courses promote math growth mindset?
RQ2	To what extent does learning-strategy instruction in gate-keeper math courses promote SRL?
RQ3	What is the nature of the association between students' math growth mindset and SRL? When and how is math growth mindset consequential for SRL and vice versa?
RQ4	To what extent do learning-strategy instruction, math growth mindset, and SRL predict students' performance in gate-keeper math courses?

Methodology

Intervention

We used various types of activities to inherently integrate learning strategies within four gate-keeper math courses to promote math growth mindset and SRL simultaneously. We focused on four key learning strategies: Elaboration (cognitive), Self-testing and Adaptation of Learning Approach (metacognitive), and Effort Management (management). As found by Donker et al. (2014), elaboration strategies can be effective for learning math as they help students form internal connections between existing knowledge and new material. Instructors can train students to use elaboration strategies by encouraging student explanation, sense-making, and justification using class discussions and discussion board assignments. Such discussions allow students to form a math growth mindset (e.g., Sun, 2015, p.37) and directly connect to the self-reflection phase of Zimmerman's SRL model (Zimmerman, 2000). Self-testing and adaption of learning approach strategies are two learning strategies that connect with both math growth mindset and SRL. By frequently encouraging self-testing and allowing for multiple attempts, instructors can help students develop a math growth mindset (e.g., Sun, 2015) and allow them to practice self-monitoring (the performance phase of SRL). Presenting mathematical tasks that allow for multiple solutions sends growth mindset messages and motivates students to adjust their learning strategies for better task performance (the self-reflection phase of SRL). Finally, instructors who frequently make effort attributions about math tasks encourage students to practice effort management strategies (forethought phase of SRL) and promote math growth mindset. Students were exposed to these learning strategies via a series of discussion board assignments followed by online and in-class reflections. For example, in week three students were assigned to watch a video about the study cycle and make a post and at least one reply in the discussion forum about it. In the following week, the study cycle was demonstrated by the instructor in the context of the math course and connections were made with the students' discussions from the online forum.

Study Design

The study utilizes a repeated-measures between-subjects design where four sections in each of the four target math courses (College Algebra I/II and Calculus I/II) at a large HBCU were assigned to a treatment group (2 sections) or a control group (2 sections). *Treatment* students were taught about effective math learning strategies including elaboration, self-testing, effort and time management, and test-taking strategies in the form of class discussions and activities, discussion board assignments, and short videos and quizzes. *Control* students, on the other hand, were taught the same course content without any instruction on learning strategies.

Data and Scales

To evaluate the effectiveness of integrating learning strategies into gatekeeper math courses, we use data collected during the 2022-2023 academic year from students at the study institution. The data comes from 32 sections (16 treatment and 16 control) spanning four introductory math courses. The data consists of i) students' responses to pre- and post-surveys about their math mindset, SRL, and math, gender, and racial identities; ii) students' scores on pre- and post-tests related to course content, and iii) students' demographic (gender, PELL status, and residency) and academic (STEM status, classification, and GPA) characteristics. A total of 986 students (502 control and 484 treatment) completed the pre- and post-content tests and 551 students (278 control and 273 treatment) completed the pre- and post-surveys.

Hocker's (2017) modified math mindset scale was adapted and used for measuring students' math mindsets. The Self-Regulation Strategy Inventory–Self-Report, developed by Cleary (2006), was used to measure students' SRL. The original SRL scale, validated on a sample of high school students, had three subscales: (a) Managing Environment and Behavior, (b) Maladaptive Regulatory Behaviors, and (c) Seeking and Learning Information. Racial identity was measured using Sellers et al.'s (1997) Multidimensional Inventory of Black Identity (MIBI) for Black students and Brown et al.'s (2014) Multigroup Ethnic Identity Measure for non-Black students. Gender identity was measured using a modified version of the MIBI scale. Finally, math identity was measured using Lock et al.'s (2013) math identity scale which consists of three subscales: (a) math competency, (b) math recognition, and (c) math interest.

Statistical Analysis

The data analysis included descriptive statistics, psychometric analyses of the mindset, SRL, and various identity scales, correlations, and regression modeling. All analyses were conducted using the open-source statistical software R version 4.1.3. A 5% significance level is used.

Results

The statistics in Table 2 show reasonable similarity between students in the control and treatment sections in terms of their background characteristics.

Table 2

Characteristics of the sample participants by their role in the study.

Variable	Control: n (%)	Treatment: n (%)
Gender: Female	152 (65.52%)	162 (71.78%)
STEM: Yes	128 (55.17%)	100 (44.25%)

Variable	Control: n (%)	Treatment: n (%)
PELL: Yes	208 (89.66%)	199 (88.05%)
Residency: Out-of-State	121 (52.16%)	105 (46.46%)
GPA: >= 3.00	93 (56.71%)	107 (68.15%)

The psychometric analysis, consisting of both exploratory and confirmatory factor analyses, of the mindset items showed a good fit for the mindset scale to the data: root mean square of the residuals (RMSR) = .002 for pre-survey & .001 for post-survey. On the other hand, the SRL items did not fit the original three-factor structure with the items of the Seeking and Learning Information subscale not loading on a single factor as hypothesized. The two-factor structure provided an acceptable fit with the Managing Environment and Behavior and the Maladaptive Regulatory Behaviors subscales forming two separate factors (presurvey RMSR = .056 & post-survey = .057). These two subscales, labeled *SRL-1* and *SRL-2*, did not load on a common higher -order factor (pre-survey factor cor. = -.03 & post-survey = -.18) and were analyzed separately.

Table 3

Estimates of regression coefficients (standard errors) from four regression models with the response variable shown in the column and explanatory variables shown in the rows.

Explanatory Variable	Mindset Diff	SRL-1Diff	SRL-2 Diff	Performance Diff
	Estimate (S.E.)	Estimate (S.E.)	Estimate (S.E.)	Estimate (S.E.)
Role: Treatment	0.180 (0.106) ·	0.023 (0.058)	-0.083 (0.074)	-3.924 (2.418)
Mindset Difference	-	-0.003 (0.032)	0.155 (0.040) ***	2.834 (1.348) *
SRL-1 Difference	-0.010 (0.107)	-	0.172 (0.074) *	3.281 (2.414)
SRL-2 Difference	0.319 (0.082) ***	0.107 (0.046) *	-	4.438 (1.891) *
Gen Identity Reflection	-0.100 (0.046) *	0.058 (0.025) *	-0.003 (0.032)	0.588 (1.070)
Gen Identity Centrality	0.069 (0.045)	-0.008 (0.025)	0.042 (0.032)	0.503 (1.028)
Racial Identity	0.173 (0.073) *	-0.028 (0.040)	-0.050 (0.051)	-1.932 (1.665)
Math Iden: Competency	0.063 (0.115)	-0.068 (0.063)	-0.121 (0.080)	3.265 (2.619)
Math Iden: Recognition	0.148 (0.072) *	0.013 (0.400)	-0.023 (0.050)	1.524 (1.171)
Math Iden: Interest	-0.178 (0.016) **	0.000 (0.034)	0.093 (0.043) *	-1.450 (1.429)
Gender: Male	0.136 (0.126)	0.046 (0.069)	-0.042 (0.088)	-5.338 (2.880) ·
STEM: Yes	-0.120 (0.125)	-0.053 (0.069)	-0.040 (0.087)	-1.983 (2.861)
GPA	0.014 (0.089)	0.034 (0.049)	-0.009 (0.062)	1.456 (2.003)
PELL: Yes	0.243 (0.178)	0.165 (0.098) ·	-0.090 (0.124)	1.462 (3.665)
Residency: Out-of-State	0.052 (0.110)	-0.031 (0.060)	0.059 (0.076)	-0.699 (2.498)
Class: Sophomore	-0.012 (0.116)	0.004 (0.064)	-0.118 (0.081)	-0.432 (2.646)
Class: Junior	-0.282 (0.242)	0.062 (0.133)	0.166 (0.169)	-4.430 (5.683)
Class: Senior	-0.506 (0.940)	-1.115 (0.513) *	-0.360 (0.655)	-7.612 (20.76)
Course: Algebra II	-0.111 (0.138)	0.021 (0.076)	0.043 (0.096)	8.757 (3.097) **
Course: Calc I	0.081 (0.166)	0.102 (0.091)	0.003 (0.115)	-8.352 (3.765) *
Course: Calc II	-0.003 (0.181)	0.040 (0.099)	0.313 (0.125) *	18.017 (4.166) ***
Adjusted R^2	0.078	0.014	0.088	0.212

Note: Reference category is “Control” for *Role*, “Female” for *Gender*, “No” for *STEM* and *PELL*, “In-State” for *Residency*, “Freshman” for *Classification*, and “Algebra I” for *Course*.
Significance codes: ‘***’ ≡ p-value < .001; ‘**’ ≡ p-value < .01; ‘*’ ≡ p-value < .05; ‘·’ ≡ p-value < .1

Several regression models were developed to investigate research questions **RQ1**, **RQ2**, and **RQ4**. The results are summarized in Table 3 where the response variables represented the change in math mindset (*posttest score – pretest score*), the change in *SRL-1* and *SRL-2*, and the change in performance on the content tests. These results suggest that 1) accounting for students' SRL, identities, background characteristics, and course, learning-strategy instruction was associated with positive, yet marginal (coef. = 0.18, p =.090), improvement in math growth mindset; 2) accounting for students' math mindset, identities, background characteristics, and course, learning-strategy instruction was not significantly associated with changes in SRL; 3) both gains in math mindset and gains in *SRL-2* were positively associated with gains in students' performance on the content tests (*Mindset*: coef. = 2.83, p = .036; *SRL-2*: coef. = 4.44, p = .020) but learning-strategy instruction was not significantly associated with performance gains.

Correlation analyses were conducted to test the bidirectional association in **RQ3**. The bivariate Pearson correlation analysis between students' math mindset scores and SRL scores showed that 1) students' initial mindset and SRL scores were moderately correlated with their post-semester mindset and SRL scores (*Mindset*: cor. = .53, 95% CI [.47, .59], p < .001; *SRL-1*: cor. = .69, 95% CI [.64, .73], p < .001; *SRL-2*: cor. = .50, 95% CI [.44, .56], p < .001), and 2) students' post-semester *mindset* scores had weak positive correlation with their initial *SRL-2* scores (cor. = .36, 95% CI [.28, .43], p < .001) but were not significantly correlated with their initial *SRL-1* scores (cor. = -.02, 95% CI [-.10, .07], p > .999). Additionally, the cross-lagged correlation analysis revealed that 1) students' initial math mindset was not predictive of their end-of-semester SRL (*SRL-1*: coef. = 0.004, p = .872; *SRL-2*: coef. = 0.04, p = .175) given their pre-semester SRL score, and 2) only students' initial *SRL-2* was predictive of their end-of-semester math mindset (coef. = 0.29, p < .001) given their pre-semester math mindset. These results suggest a unidirectional relationship between math mindset and SRL.

Concluding Remarks

The results reported in this study are preliminary results from one phase of data collection. We will continue to refine these results using data from additional data collection phases and revised analysis plans. For instance, we will apply the retrospective pretest-posttest (RPP) design (e.g., Little et al., 2019) using data from the 2023-2024 academic year and contrast the results with the traditional pretest-posttest design. The RPP design allows participants to gauge the degree of change that they experience with greater awareness and precision by asking respondents to rate survey items twice during the same posttest measurement occasion from two specific frames of reference: “now” and “at the start of the semester.” Furthermore, future phases of analyses will account for additional factors such as instructor and class attendance.

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SECONDARY STUDENTS' DIFFERING AFFILIATIONS WITH THEIR MATHEMATICS CLASSROOM OBLIGATIONS

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The purpose of this research study is to characterize the way two secondary students affiliate with their perceived obligations during various segments of their mathematics classroom instruction. Based on individual student interviews and lesson observations, results indicate that although the students perceived similar obligations for most of the instructional segments, they affiliated in different and interesting ways with their perceived obligations.

Students' experiences in mathematics classrooms have long been identified as important for their learning and enjoyment of mathematics. How students experience their mathematics classrooms influences the identities they form about themselves as mathematical thinkers and doers (Bieda & Staples, 2020; Boaler, 2000) and the perceptions they form about their mathematics classes and the general discipline of mathematics (Ellis et al., 2014). Students' identities and perceptions have been shown to influence their future mathematical experiences and career choices (Boaler, 2000; Ellis et al., 2014).

In this study, I present two cases of secondary school students as they discuss their mathematics classroom experiences. I describe how the students perceived their obligations during various segments of their mathematics lessons as well as how they identified (or not) with those perceived obligations. Understanding students' perceptions of their classroom obligations is important for educators as they work toward creating classroom activities that are both meaningful and enjoyable for students. And creating such classrooms will aid in the development of more positive mathematical identities among students.

Theoretical Framing and Related Literature

This study is guided by the perspective that students are participants in the social context of classroom learning (Lave, 1996). As students and their teacher(s) engage with each other during mathematics classroom learning, they participate in social practices wherein norms and obligations for what counts as mathematical competence are developed and negotiated (Cobb et al., 2009; Langer-Osuna, 2016). Cobb and colleagues discuss two kinds of identities—normative identity and personal identity— that students develop because of their classroom norms and obligations.

The normative identity refers to the communally constructed obligations for what students must do to be considered effective and competent members of the classroom community (Cobb et al., 2009; Ruef, 2021). These obligations are created and reinforced over time through implicit and explicit classroom discourse and consist of actions that students must do to fulfil the classroom community's expectations. The normative identity includes both general social norms (i.e., things students should do to be considered good and cooperative colleagues) and specifically mathematical norms (i.e., things students should do to be viewed as mathematically competent). Moreover, a classroom's normative identity is context specific (Gresalfi et al., 2009; Ruef, 2021), meaning that what is considered a "normative/competent student" in one classroom context may be different in another classroom. For example, in some classes, it may be normative for students to develop non-standard solution methods, question each other's claims, and/or be expected to justify their reasoning. In other classes, the expectation might be for students to demonstrate proficiency in employing standard algorithms and rely on the teacher to determine their correctness.

Personal identity, on the other hand, relates to if and how students identify with their classroom normative obligations. Identification involves a process whereby one turns (or not) the communally constructed expectations into what the individual thinks and says about themselves (Cobb et al., 2009). Cobb and Hodge (2007) define personal identity as "an ongoing process of being a particular kind of person in the local social world of the classroom" (p. 168). Studies on students' personal identities suggest that students' identities are not stable across instructional activities—students affiliate with some activities over others and respond to them in interesting ways (e.g., Andersson et al., 2005). To this end, the current study was guided by the following research questions:

1. What do two middle school, pre-algebra students perceive to be their obligations, both in general and specific to mathematics, during various instructional segments?
2. How do the students identify with these obligations?

Method

Context and Participants

This study is part of a larger multiple case-study (Yin, 2003), designed to examine students' perceptions of their mathematics classroom obligations. The study was conducted among secondary students at a junior high and high school in the midwestern US. For this study, I focus on two students, Amanda and Bonnie (pseudonyms) who were enrolled in the same 8th grade pre-algebra class in Spring 2023. Their pre-algebra class had a total of 15 students. I chose Amanda and Bonnie for my case study because they demonstrated different participation patterns in the class and affiliated differently with the classroom norms as evidenced in both classroom observations and their individual interviews. Despite differences in how they participated, these two students represent typical cases since their classmates also participated in differential measures during various instructional segments. Below, I provide some additional details about both Amanda and Bonnie.

Amanda self-identified as a white, transgender, 8th grader who aspires to be a graphic designer in the future. They portrayed a calm demeanor in the class and always sat quietly at their desk during whole class and individual work time. They rarely answered any of the teacher's questions nor asked any questions publicly during whole class discussions but happily discussed ideas with their peers during group work time.

Bonnie self-identified as a white, female, 8th grader who aspires to be a beautician in the future. When she walked into the class, she always socialized with her peers and went to the teacher's desk to greet him before settling at her desk. She raised her hand to answer almost every question asked by the teacher and frequently asked questions publicly during whole class discussions. She expressed that both individual and group work were okay for her because she is diligent enough to work alone and enjoys working with her peers as well.

Data Collection and Analysis

Data sources for the larger study consisted of lesson observations, student surveys, and individual student interviews. I conducted lesson observations (~ 50 minutes per lesson) twice a week for a period of six weeks—a total of 12 observations and administered two surveys. I used the classroom observation and survey data to select my two cases and to provide context for my interviews. After selection, I interviewed each of the focal students twice (~ 15 minutes per interview), asking them about their classroom obligations during various instructional practices and how they affiliated with those obligations.

The goal of my analysis for this study was to examine how each of the focal students described their classroom obligations during various segments of instruction as well as how they identified with those obligations. To do so, I relied mainly on interview data because I wanted to elevate the students' perspectives over mine (i.e., what I observed) since it is their interpretation of classroom norms and their personal identity narratives that mattered for this study. I used the other data (i.e., from classroom observations and surveys) as secondary data to clarify, contextualize, and triangulate my findings. For example, observation data showed evidence of how students demonstrated affiliation or resistance to their classroom obligations.

I analyzed the transcribed interview data in two phases. In phase one, I read through each transcript and identified instances where the students talked about specific segments of instruction. In phase two, I read through each segment and coded for what students perceived to be the normative identity as well as their personal identities based on the interpretive scheme for student identity framework (Cobb et al., 2009). I briefly describe these codes in tables 1 and 2 below. Finally, I conducted a cross-case analysis, looking across both students' data to identify emerging themes.

Table 1

Normative Identity Codes

Category	Description
General classroom norm 1	Authority figure – identifies to whom students think they are accountable. This could either be the teacher, peers, self, or all.
General classroom norm 2	Agency type – relates to how students believe they can legitimately express agency in class and include (a) conceptual agency (i.e., can choose solution methods, justify their work, assess peers' work, make mathematical connections) or (b) disciplinary agency (i.e., can only use established solution methods and follow teacher-led activities).
Math specific norms	Relates to 'for what' students believe they are accountable. That is, what they believe counts as mathematical competence.

Table 2

Personal Identity Codes

Category	Description
Identify with	Students turn obligations-to-others into obligations-to-self. That is, they see value in the classroom obligations, develop commitment to, and enjoy taking part in them.
Comply with	Obligations-to-others remain as such. That is, students see minimal or no value in the obligations and view performing those obligations as a chore or a way of fulfilling other people's expectations.
Resist	Students reject their classroom obligations. That is, they see no value or a negative value in the obligations and deny performing them

Findings

The two students discussed what they viewed as their normative identities and shared about their personal identities during five segments of class—going over homework, lesson lecture, individual student work time, group work time, and during one-on-one work time with the teacher. Although both students perceived similar normative identities during most of the instructional segments, how they identified with those norms/obligations differed.

Students' Perceived Normative Identity

Regarding their general classroom obligations, both students expressed that they saw their obligation as that of exercising disciplinary agency. Table 3 shows the specific roles that the students believed they were obliged to perform during various segments of instruction. All these obligations lie under disciplinary agency in the framework because they involve following teacher-led activities and reproducing solution methods established by the teacher.

Table 3

Students' Perceived General Classroom Obligations

Instructional practice	Perceived Student Obligations
Going over homework	1. Ask questions about problems you did not understand. 2. Follow along while the teacher demonstrates how to solve the previous homework problems. 3. Fix answers to the problems you got wrong.
Lesson lecture	1. Listen, take notes, and ask questions to understand the solution methods demonstrated by the teacher. 2. Respond to teacher's questions to demonstrate an understanding of the solution methods.
Student individual and group work time	1. Employ the solution strategies demonstrated by the teacher to solve assigned tasks. 2. Compare answers with peers and/or check with the teacher for correctness.
One-on-one work time with the teacher	1. Ask for teacher's help on hard or confusing problems. 2. Stay focused and follow the teacher along to understand as he shows you how to solve the problems. 3. Respond to the teacher's questions.

Moreover, both students viewed their teacher as the main authority during most segments of class with a few exceptions. According to them, the teacher decided what topic and questions the class would focus on and determined what counts as a correct answer. There were however two exceptions noted by both or one of the students. First, both students mentioned that whenever the class was going over previous homework, they could share in the authority of deciding what questions the class should focus on. However, the two students had differing perceptions about the classroom authority dynamics whenever they were working individually and/or in small groups. On the one hand, Amanda viewed the teacher as the sole authority for determining the legitimacy of their individual and their peers' solutions, an aspect that appealed to them as demonstrated by their statement below:

> [the teacher] usually tells us like, yeah, you're on the right track or close or no, you're off a lot. And I appreciate that because sometimes if I'm like working by myself, I don't know if I'm on, like, even close to getting any of the questions right.

On the other hand, Bonnie viewed the authority structures as hierarchical, with the teacher being at the top of the hierarchy and her and her peers being lower in the hierarchy. Although she viewed the teacher as being, "always right, because he's like a walking calculator," Bonnie believed that she could determine the correctness of her solutions for questions that were simple, but relied on the teacher's authority for harder problems. She said that she used the one-on-one work time with the teacher to confirm her correctness on harder problems.

Regarding their specifically mathematical obligations, both students described the normative identity of a competent and successful mathematics student as one who gets the correct answers and consequently good grades. As such, both students said that they always strive towards getting the right answer whether they are working individually, in small groups, or with the whole class. Amanda added that whenever the teacher assigns tasks, they, alongside their peers, believe they should "work them out to get the right answer …and show how we got our answer." They believed that they were obliged to clearly outline the steps for getting the right answer. As mentioned earlier, these steps would have already been demonstrated by the teacher, and it was mostly the teacher who determined if an answer was correct. Similarly, Bonnie expressed her goal as getting correct answers when working individually. However, unlike Amanda who saw group work as a space for them and their peers to collaboratively work towards getting the right answer, Bonnie saw group work as a space to show her peers how to solve the assigned tasks as demonstrated in her quote, “In my group, I pretty much just do the math, because I pay attention, like the most. I work out everything more than most people. So, I just do the math and tell them the answers.” For Bonnie, getting correct answers is important and she believed in her ability to get those answers. She perceived herself as better in math than most of her peers (see more in the section below) and believed she had to “do the math and tell [her peers] the answers.”

Beyond articulating their perceived normative identities, Amanda and Bonnie expressed contrasting views regarding how they self-identified with the norms. I now turn to how the students identified with their normative identity.

Students’ Differing Personal Identities

The two students differed in the ways they talked about themselves with respect to the normative identities. Bonnie viewed herself as a model student and identified with most of the obligations she perceived for the various segments of instruction. She said her favorite segment of class was solving tasks individually, although she did not mind doing groupwork because “...if someone needs help, I'm like, willing to help them.” Additionally, she described herself as a competent student. She said the following about herself, “I'm a good student in math and any of my classes, because I'm very calming content, and I focus good. And I'm a fast learner, and I'll do what it takes to get a good grade.” Bonnie saw herself as a competent student who meets the criteria for competency that she and Amanda had separately mentioned earlier. That is, getting good grades and getting them fast.

The only segment that she expressed a dislike for was the lesson lecture, citing that, “it's kind of boring. Because he’s [the teacher] just talking.” Despite her dislike, Bonnie complied with her perceived obligations during the lesson lecture by taking notes and “if it comes to answering question, then I'll get involved.” I concluded that Bonnie not only perceived herself as a good student, but she also performed all of her perceived obligations whether she *identified with* them (e.g., solving tasks individually and in groups) or simply *complied with* them (e.g., taking notes and listening to the teacher while “he’s just talking.”)

Amanda, on the other hand, described themself as an “average student” and mentioned Bonnie when I asked if they could think of someone in their class who is good in math. They expressed that they do not enjoy the classroom norms of taking notes, asking the teacher questions, and responding to the teacher’s questions during lecture time and when going over homework. They added that they prefer working with peers in small groups rather than working individually because, “it just like makes more sense. And we can like bounce off each other's ideas.” During my lesson observations, I noticed Amanda resisting these norms by sitting quietly and drawing in their book while most of their peers were taking notes during the lecture and review of previous homework. Interestingly, Amanda would become lively and happily engage in discussions with their peers whenever they started doing group work. I probed further to know why group work was the only segment of class that Amanda identified with. They said the following:

> I prefer the groups because I feel like the way he teaches it, it's harder, because people are always interrupting him and stuff. I feel like when I'm in a group, I can understand what's happening easier because we are closer together, and it is less distractions.

For Amanda, group work was the only segment of class they *identified with* because it was structured in a way to support their mathematics learning. According to them, the other segments (e.g., lesson lecture and going over homework) were full of distractions, making it hard for them to follow through. Elsewhere, Amanda said that individual work time felt isolating. Notice that unlike Bonnie who would comply with segments of class that she didn’t like, Amanda *resisted* the segments they disliked by shutting down, not participating, and doing alternative things (e.g., drawing on their book rather than taking notes).

Discussion and Conclusion

In this study, I examined how secondary students perceive and affiliate with their mathematics classroom obligations. I found that my two case study students do perceive similar obligations for most of their classroom instructional segments, but they differ in how they affiliate with those obligations. Taken together, the students’ personal identity narratives revealed that they identified with the obligations that they not only enjoyed, but also found important for their success and that of their peers.

This study contributes to various important discussions within and beyond the mathematics education field. First, it demonstrates the importance of welcoming and listening to students’ voices about their classroom experiences and identity narratives. Oftentimes, teachers create narratives about their students based on the students’ task engagement (see Aaron & Herbst, 2015). However, if an educator only observes Amanda’s behavior and engagement levels and doesn’t go the extra mile to hear Amanda’s perspective, the educator may easily define Amanda as defiant or a math hater. Moreover, an educator who doesn’t understand Amanda’s value for groupwork and who doesn't create opportunities for groupwork in their class may make math learning an unpleasant experience for Amanda.

Second, this study adds to the larger discussion about students' views on their classroom authority and agency, and its impact on their identity formation. The current findings show that although these two students, just like those reported in other studies (e.g., Amit & Fried, 2005; Webel, 2010) viewed the teacher as the ultimate authority, they exercised their agency in different ways and affiliated differently with their classroom norms (i.e., by either identifying, complying, or resisting). Therefore, the question is not whether students *have* or *lack* agency because, "even the most constrained are able to exercise agency at the very basic level, by complying or resisting" (Gresalfi et. al., 2009, p. 53). Rather, the question is *what kinds of agency* do various students exercise in their mathematics classrooms, and how does their form of agency support their learning, that of their peers, and their process of identity formation? Future research could explore how to (a) support students to develop positive mathematical identities and (b) support teachers to welcome their students' voices and work with them to positively shape classroom norms.

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Framing Elementary Mathematics through Problem-Solving, Reasoning, and Fluency

LAUNCHING: SETTING THE STAGE FOR MATHEMATICAL SENSEMAKING

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The launch (or introduction) of a task is a short, but important, phase of problem-solving lessons. We explored how 35 elementary school teachers launched fraction story problems to support children's sensemaking during problem solving. Findings identified three common elements of launches: (a) explaining the task components, (b) fostering children's visualization of the story, and (c) connecting the story to children's experiences. Our study addresses the field's call for images of launches, enhances the limited work on launching with elementary-aged children, and provides some specific language for discussing launching. Also showcased are opportunities unique to launching story problems.

Launching is the focus of a small, but growing, body of research that seeks to understand how teachers introduce mathematical tasks in ways that support children's sensemaking during problem solving (National Council of Teachers of Mathematics [NCTM], 2014). In the launch phase of a lesson—also called "task set-up," "unpacking," or "problem posing"—teachers have opportunities to introduce tasks in ways that jumpstart children's sensemaking, so they are ready to begin and persist in problem solving (Carpenter et al., 2015; Stein et al. 1996).

Launching received widespread attention with Jackson and colleagues' (2013) seminal work, which found an association between high-quality launches and greater student learning opportunities during whole-class discussions that showcased students' solution strategies. Launches were characterized as high-quality when they helped the class develop shared understandings of tasks through discussion of key mathematical features and, for story problems, key contextual relationships, all while maintaining the cognitive demand of the tasks. Jackson and colleagues also put forth ideas for future research, including the development of a "set of images that could serve as a foundation for 'describing the implicit grammar' of setting up tasks" (p. 679). Others have echoed this call for images not only to enhance the field's understanding of launching but also to support teachers' enactment. For instance, Tyminski and colleagues (2019) argued that teachers have struggled to craft high-quality launches, in part, because of the field's lack of clarity about what teachers say and do during these launches.

A decade after Jackson and colleagues' (2013) study, this need still exists and is especially acute at the elementary level, given that most launching research has been at the secondary level or with prospective teachers (see, e.g., González & Eli, 2017; Parrish et al., 2023). Our study answers this call by identifying and illustrating common elements of launches with elementary-aged children. We also restricted our study to story problems, given their well-researched connections to mathematical sensemaking (Carpenter et al., 2015). Our goal was not to identify the "best" elements, but instead to begin to map the landscape of possible elements for launches to support children's sensemaking. We explored this research question in the domain of fractions: What were the common elements of launches used by elementary school teachers to support children's sensemaking with story problems?

Methods

Data were drawn from a larger project, *Responsive Teaching in Elementary Mathematics* (RTEM), in which teachers engaged in up to three years of professional development focused on fraction teaching that centers children's sensemaking and is responsive to their ideas (Empson & Jacobs, 2021). Here, we analyzed video (and corresponding transcripts) of 35 launches. We operationalized launches as beginning when teachers used a transitional phrase (e.g., "let's begin") or projected a story problem or related image on the board. We considered launches to have ended when children were released to begin working on the problem or when teachers transitioned to speaking to children one-on-one.

Participants

We explored one launch per teacher for 35 teachers of grades 3–5 (30 females, 5 males). Teachers varied in their instructional contexts and professional experiences, which increased the likelihood that we would see variety in teachers' launches. Specifically, teachers were drawn from 25 schools in 3 demographically diverse neighboring districts in the southern United States, and their teaching experience ranged from 2–36 years (M = 12.3 years). Launching data were collected during a single school year, when teachers were at the end of their first (N = 9), second (N = 15), or third (N = 11) year of the RTEM 3-year professional development.

Data Sources

The 35 launches were drawn from a set of video-recorded lessons collected as part of the RTEM project. In these lessons, teachers all launched the same type of fraction story problem—an equal sharing problem with a fractional answer, such as *6 children sharing 10 pancakes equally*—but each teacher chose the problem context and numbers appropriate for their class. We selected the 35 launches for analysis based on three criteria. First, we focused on launches in which the teacher worked with the whole class (vs. small groups). Second, we focused on launches in which we had evidence (from earlier analyses) that children's sensemaking was visible and valued in the circulating or discussion phases of the lesson (Empson & Jacobs, 2021). As such, we were able to examine how launches might have set the stage for sensemaking. Third, we focused on launches in which the cognitive demand of the task was maintained, as recommended by Jackson and colleagues (2013). Specifically, teachers did not suggest solution strategies or alter original tasks to make them less challenging during the launches.

Analysis

Analysis involved a constant comparative process to uncover patterns in teachers' launches (Corbin & Strauss, 2014). We began with open coding of both video and transcript data, and then iteratively refined our codes to identify what we called a "launching element." A *launching element* was a section of a launch that supported children's sensemaking in a particular way (e.g., visualizing the story). Launching elements varied in length, and a single launching element often included multiple teaching moves and interactions, but there was coherence in the way sensemaking was supported. All data were double-coded, and interrater reliability was consistently 85% or higher, with discrepancies resolved through discussion. Basic descriptive information for each launch (e.g., launch length and classroom organization) was also tracked.

Findings

We found that launches were generally short, ranging from 0.5 to 8.0 minutes (M = 3.4 minutes), but the organization of classrooms varied. For instance, we found variety in terms of children's physical location during the launch (e.g., seated on the carpet or at desks), problem presentation (e.g., paper or whiteboard), class participation norms (e.g., hand raising or open conversation), and communication structures (e.g., whole-class discussion or turn-and-talk). The short duration of launches and the variety in classroom organization suggest that launching could fit into most lessons and teaching styles.

The content of launches generally included not only the introduction of the equal sharing problem, but also instructions for the problem-solving time following the launch (e.g., location for finished work or encouragement to solve the problem in multiple ways). In this paper, we focus on the introduction of the story problem, and we identified three common elements of launches: (a) explaining the task components, (b) fostering children's visualization of the story, and (c) connecting the story to children's experiences. In a single launch, teachers sometimes used only one element and other times used two or three elements in combination.

In the following sections, we provide an overview and an illustrative example for each launching element. To aid the reader in distinguishing launching elements, we selected examples that (a) clearly illustrate a single launching element and (b) focus on sensemaking with the same two components of equal sharing problems: items to be shared and sharing equally.

Explaining the Task Components

For this launching element, teachers supported children's sensemaking by directly explaining the task components as they were written. At times, teachers simply elicited or restated task components, such as the number of sharers or the number of items to be shared. Other times, teachers provided definitions or examples to clarify task components.

Example. In this 4-minute launch, Ms. N drew her third graders' attention to important components of the story problem in which she was named as a character. In Table 1, we join the launch near the beginning, immediately after Ms. N had read the problem aloud.

Table 1

Ms. N Launches by Explaining the Task Components

Problem: *Ms. N baked 6 delicious strawberry cakes for her 4 friends. If the cake is shared equally, how much cake will each friend get?*	
Ms. N:	So, six cakes and I'm sharing with four friends. Okay. How many cakes?
Children:	Six.
Ms. N:	Six. And I'm sharing with how many?
Children:	Four.
Ms. N:	Four friends. Okay? I don't need any. I'm just sharing with four people. And we are going to share equally. Let's talk about equally. What does that mean if we're sharing something equally? Oliver.
Oliver:	It has the same number, amount.
Ms. N:	Same number, same amount. Right. So, if you're sharing equally and somebody got 50 and someone else got 20, is that equal?
Children:	No.
Ms. N:	No. Okay, what about 10 and 10? Would that be equal?
Children:	Yes.
Ms. N:	Perfect.

In this excerpt, Ms. N took a step-by-step approach to addressing task components. She first drew children's attention to the number of cakes and friends, clarifying that she was not part of the group of sharers. She then supported children in understanding the meaning of sharing equally. Specifically, she elicited and restated Oliver's definition of equal sharing as "same number" (or "same amount") and then asked children to consider examples (e.g., 10 and 10) and non-examples (e.g., 50 and 20). These efforts to draw attention to and clarify important task components can encourage children to reason about these components when engaging with the written tasks during problem solving.

Fostering Children's Visualization of the Story

For this launching element, teachers supported children's sensemaking by fostering their visualization of the story. Sometimes teachers displayed story-related images (e.g., a picture of the story setting). Other times, teachers used words to foster visualization. For instance, they embellished the story by adding details about the items to be shared, suggested children imagine themselves as characters in the story, or invited children to "picture" aspects of the story.

Example. In this 4.5-minute launch, Ms. T engaged her fifth graders' imaginations by inviting them to visualize submarine sandwiches and sharing them equally. In Table 2, we join the launch near the end, after Ms. T had read the problem aloud and reviewed the quantities.

Table 2

Ms. T Launches by Fostering Children's Visualization of the Story

Problem: *4 children want to share 10 submarine sandwiches so that everyone gets the same amount. How much can each child have?*	
Ms. T:	So, when we're talking about submarine sandwiches, is there something you're picturing in your head by any chance? [*calling on Elise*] Yes.
Elise:	Foot-long from Subway.
Ms. T:	Foot-long from Subway. Yes. Okay. What about you?
Colton:	Like this PES movie, like the PES one—
Ms. T:	Oh yes. The PES video—when they made the submarine sandwich out of all the stuff. That was awesome! *["Submarine Sandwich" is a stop-motion short film by PES (pesfilm.com)*] Have you guys seen the big, long party subs?
Children:	Yeah.
Ms. T:	They're six feet long… I don't think any of them could probably eat the six-foot [sandwich], even with four children. So, I think the foot-long sub is a good thing. How many of you have seen a foot-long sub from Subway or somewhere like that?
Many children raise their hands.	
Ms. T:	Okay, so that's what we're looking at. We have four children, maybe these four people right here *[points to four children],* and they want to share. I'm bringing them 10 sub sandwiches for being so awesome on their last math test, and they start splitting that up. Okay, how would they do it to where all of them get the same amount? Because Jacob is going to be really upset if Derrick gets more than him. He's going to be like, "Excuse me. No, that's not happening. He gets an equal amount. We both did well. We're getting the same amount!"

In this excerpt, Ms. T elicited what children imagined when they heard "submarine sandwiches." Children offered various mental pictures, such as foot-long sandwiches from Subway and memorable video scenes. Ms. T then illustrated sharing equally by using specific children in the classroom as characters in the story. In doing so, she supported children in envisioning the story in a way that was meaningful to them, which provided space for them to imagine potential concerns (like Jacob's feelings) if the sandwiches were not shared equally. These ways of using children's imaginations to visualize the story can encourage children to "see" the story in their minds in vivid and meaningful ways.

Connecting the Story to Children's Experiences

For this launching element, teachers supported children's sensemaking by connecting the story to their lived experiences. At times, teachers made connections to children's personal experiences outside of school, as in the example below. Other times, teachers made connections to shared experiences of the class. For instance, teachers cued children's memories of past class celebrations, school events, or class problem-solving activities that connected to the story.

Example. In this 5.5-minute launch, Ms. L elicited her third graders' experiences with eating and sharing cookies, as in the story. In Table 3, we join midway in the launch, after Ms. L had given some instructions for group work, but before she had introduced the problem itself.

Table 3

Ms. L Launches by Connecting the Story to Children's Experiences

Problem: *2 children are sharing 11 cookies. If they want everyone to have the same amount, how much should each child get?*	
Ms. L:	Someone tell me what's your most favorite kind of cookie? Most favorite kind. I have to think about that because I love all cookies. Lucas, what's your favorite?
Lucas:	Chocolate chip.
Ms. L:	Chocolate chip! So like the picture that I have. Ellie, what's your favorite?
Ellie:	Sugar cookies!
Ms. L:	Ooh, do you like with icing or no icing?
Ellie:	Both.
Ms. L:	Ooh. Yum, yum, yum… [discussion of favorite cookies continues for some time] Have you ever had to share cookies?
Children:	Yeah.
Ms. L:	Can you raise your hand and tell me a time when maybe you had to share cookies with someone? Angel?
Angel:	When I was with my brother when we went to Mexico…
Ms. L:	You had to share cookies when you went to Mexico?
Angel:	Yeah.
Ms. L:	Why? Can you tell me the story?
Angel:	Because there was only one more cookie left on the plate…and my brothers had to share, and they took it all.
Ms. L:	Oh no…so you didn't get an equal amount? How did you feel when that happened?
Angel:	It always happens!

Ms. L:	Oh, because you're used to it…So anyone else have to share cookies with somebody? Gael?
Gael:	Once I was at my house, my cousin Maria, she ate all the cookies. She ate almost all the cookies. There was one left. And then I had to share it with my sister because she didn't have a cookie, and I broke it in half and then I give it to her.
Ms. L:	So, when we share cookies, what's really important? When we share anything, what's really important? Raise your hand. What's really important, Camila?
Camila:	Equal amount.
Ms. L:	Why is that important?
Camila:	Because if you have a pizza and it's not shared equally, someone might cry.
Ms. L:	Yeah, because why would they cry?
Camila:	Because they only have—
Aron:	You'll get the little one.
Ms. L:	Right. You don't want to get the little piece!

In this excerpt, Ms. L elicited children's favorite cookies and experiences sharing cookies, which included unequal sharing. She then drew upon these personal experiences to help children articulate the need for sharing equally. For instance, Camila explained someone might cry if they did not get an equal amount. These types of connections between the story and children's lived experiences can encourage children to use their own experiences as problem-solving resources.

Discussion

We identified and illustrated three common elements of launches to support children's sensemaking during problem solving. In doing so, we address several gaps in the literature. First, by providing detailed descriptions of what teachers said and did during launches, we begin to answer the field's call for a set of images of launches (Jackson et al., 2013). Second, by focusing on launches of elementary school teachers, we enhance the field's understanding of launching with elementary-aged children, an under-researched group in this literature. Third, by introducing three common launching elements, we offer specific language for discussing launching, which improves the field's limited shared technical language (Wieman, 2019).

Our findings also extend prior work on launching by foregrounding some unique opportunities available when launching *story problems*. Specifically, teachers leveraged the storytelling nature of the task with two of our launching elements—fostering children's visualization of the story and connecting the story to children's experiences. Storytelling is an engaging form of communication that appeals to emotions, and the power of stories in helping children make sense of and gain access to mathematical ideas is well documented (Carpenter et al., 2015; NCTM, 2014). Further, emphasizing storytelling provides teachers with opportunities to elevate children's funds of knowledge (Tekkumru-Kisa et al., 2020). For instance, when teachers connect the story to children's lived experiences, children are explicitly invited to use their family and community experiences as resources. Our findings advance the field's understanding, but future research needs to explore launching with other types of story problems, beyond the equal sharing problems discussed here. We close with an appreciation for the richness and creativity of the teachers' launches to support children's sensemaking, especially given the often-unrealized potential of this short, but important, phase of problem-solving lessons.

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TAKING UP SPACE IN CLASSROOM INTERACTIONS

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*This study examined the quality of children's fraction learning opportunities during the circulating phase of instruction in Grades 3, 4, and 5 (*N*=185), where teachers had one-on-one conversations with individual children about their mathematical thinking. I used the construct taking up space to conceptualize an expansive view of children's fraction sensemaking, including spoken responses, gestures, and written work. Analysis revealed that children took up space to a great or limited extent in a majority of interactions, suggesting that one-on-one conversations about children's strategies for fraction story problems contribute to children's opportunities to learn.*

Learning Opportunities in One-on-One Conversations

In a series of editorials, Cai and colleagues (Cai et al., 2017; 2020) urged educational researchers to "consider how to best create the learning opportunities needed to maximize the impact on students' learning" (Cai et al., 2017, p. 234). Past research has suggested that instruction can open or constrain individual children's opportunities to learn (Franke et al., 2009; Munson, 2019). Researchers have documented how whole-group discussions (Gresalfi, 2009; Webb et al., 2014) and small-group work (Cohen & Lotan, 1997; Yackel et al., 1991) can contribute to such opportunities; however, there has been little attention to other parts of instruction.

This study focused on an under-researched part of instruction in which the teacher circulates while children work to solve a problem and have one-on-one conversations with individual children (Lindfors-Navarro, 2023). Using the construct *taking up space,* adapted from Hand (2012) and Johnson (2017), I examined the quality of children's opportunities to learn fractions during these conversations via their agentic mathematical activity—in the form of problem-solving and articulating that problem-solving. As children made sense of fractions through their spoken responses, gestures, and written work in expansive ways, they take up space. For instance, when problem-solving, children articulated their reasoning, drew pictures showing how items were partitioned and distributed, and used gestures as they described cutting or distributing shares with others (Lindfors-Navarro, 2023). Taking up space addresses children's involvement in opportunities to learn and is defined as expansive participation in mathematical sensemaking in a way that involves children's agency. For this paper, I focus on the following research question: To what extent do children take up space in one-on-one circulating interactions with a teacher involving a fraction story problem?

Methods

This study drew on data from the Responsive Teaching in Elementary Mathematics project (Jacobs et al., 2019). One lesson was examined from classrooms in Grades 3, 4, and 5 with teachers who demonstrated some skill at questioning in ways that were responsive to children's mathematical thinking (Empson et al., 2022; Jacobs & Empson, 2016). Each video-taped lesson centered a task that consisted of at least one equal sharing story problem (Empson & Levi, 2011). A total of 20 teachers and 145 children were included, drawn from schools that represented socio-economically and racially diverse populations. With this refined data set, I identified 185 one-on-one interactions taking place during the circulating phase of instruction.

Analysis

Analysis began with identifying one-on-one interactions between a teacher and individual children about children's mathematical thinking during the circulating phase of instruction. Next, I organized interactions into four scenarios that described children's progress in problem-solving at the beginning of an interaction (see Table 1). I coded and analyzed videotaped interactions, not transcriptions, and children's written work to leverage a multimodal approach (Kress, 2010) to gather evidence of taking up space.

Table 1

*Initial Problem-Solving Progress Scenario and Taking Up Space (*N *= 185)*

Scenario	Description	Number (Percent)
A	Child has completed a valid strategy and has a correct answer.	54 (29%)
B	Child has completed a valid strategy but does not yet have a correct answer.	37 (20%)
C	Child has not yet completed a valid strategy but is working towards one.	49 (26%)
D	Child has not yet completed a valid strategy and is not clearly working towards one.	45 (24%)

Note: These scenarios describe how far along a child was in their problem-solving at the beginning of the interaction

I identified dimensions involved in children taking up space as they engaged in and reflected on fraction problem-solving. All interactions during the circulating phase (*N* =185) were scored holistically for the extent to which children took up space with fractions, which included to a great extent, a limited extent, or little to no extent using a three-point scale corresponding to scores of 3, 2, and 1. This scoring is consistent with prior research on children's participation in problem-solving and articulation of their problem-solving (Gearhart et al., 1999).

Findings and Discussion

Analyses revealed that children took up space to a great (N = 70) or limited extent (N = 71) in a majority of the 185 interactions, suggesting that one-on-one conversations about children's strategies for story problems contribute to children's opportunities to learn. The conditions for taking up space with fractions included a story problem and teaching that made space for children's mathematical thinking.

Taking Up Space

The two main dimensions that distinguished taking up space in one-on-one interactions involving fractions included the extent to which children (1) demonstrated in-chargeness and (2) engaged in mathematical activity via problem-solving and articulation of problem-solving.

In-Chargeness

The dimension of in-chargeness involves children making mathematics their own, a critical component for learning with understanding. When in charge, children tend to have the originating ideas for sensemaking and exhibit a sense of investment and ownership of the mathematical ideas (Carpenter & Lehrer, 1999).

Mathematical Activity: Problem-Solving and Articulation of Problem-Solving

Children's mathematical activity could be a mix of problem-solving and articulation of problem-solving depending on children's initial problem-solving progress at the beginning of an interaction. Problem-solving and articulation of problem-solving are complementary mathematical activities and support learning (Webb et al., 2014). When children are engaged in problem-solving, they are solving a problem for which they do not already have an immediate solution. Articulation involves reflecting on their strategies for problem-solving and how they determined a solution (Ing et al., 2015). Children can articulate their problem-solving while working on their strategy or after completing a strategy. To showcase what taking up space to a great extent looks like in one-on-one interactions, I present one carefully selected interaction of a child who took up space to a great extent.

Taking Up Space to a Great Extent

When children took up space to a great extent, they problem solved by making and enacting their own problem-solving decisions, and they articulated their problem-solving by including most, if not all, the key fraction problem-solving decisions.

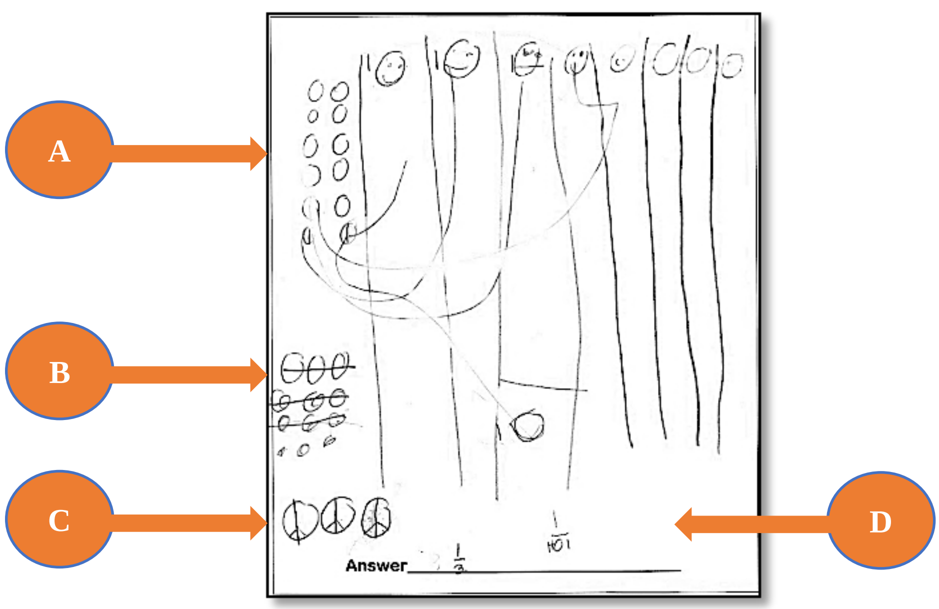
Isaac

Isaac, a third grader, was working on the problem, *9 children want to share 12 small cakes. How much cake can each person have if they all get the same amount?* Before the interaction began, Isaac had completed a valid strategy with an incorrect answer of "3", so an initial problem-solving scenario B (see Table 1).

Isaac's written work in Figure 1 shows that he had drawn the 9 children across the top of the page (with the ninth child placed further down the page) and the 12 small cakes on the upper left side of the page in six groups of two. He recognized that each child would get one whole cake and that he needed to partition the remaining three cakes to share them equally among the nine children. Isaac first tried partitioning the remaining cakes into halves. When he had distributed

Figure 1

Isaac's Written Work for 9 Children Equally Sharing 12 Small Cakes



the pieces by drawing a line from each piece to a sharer (see part A in Figure 1), he saw that he did not have enough pieces for everyone. He then tried fourths by counting 4, 8, 12 on the three extra cakes and saw that he would have more pieces than children. Finally, Isaac redrew all of the cakes (below the first set of cakes on the left side of his written work, see part B in Figure 1), crossed out the 9 whole cakes that he wanted to give to the 9 children, and partitioned each of the remaining into thirds (see part C in Figure 1). Based on this work, he had initially concluded that the answer was 3.

Isaac's written work in Figure 1 shows that he had drawn the 9 children across the top of the page (with the ninth child placed further down the page) and the 12 small cakes on the upper left side of the page in six groups of two. He recognized that each child would get one whole cake and that he needed to partition the remaining three cakes to share them equally among the nine children. Isaac first tried partitioning the remaining cakes into halves. When he distributed the pieces by drawing a line from each piece to a sharer (see part A in Figure 1), he saw that he did not have enough pieces for everyone. He then tried fourths by counting 4, 8, 12 on the three extra cakes and saw that he would have more pieces than children. Finally, Isaac redrew all the cakes (below the first set of cakes on the left side of his written work, see part B in Figure 1), crossed out the 9 whole cakes that he wanted to give to the 9 children, and partitioned each of the remaining into thirds (see part C in Figure 1). Based on this work, he had initially concluded that the answer was 3.

The interaction began with Isaac beckoning the teacher with a raised hand. She approached and Isaac began to explain his thinking (see Table 2):

Table 2

Isaac's Explanation of His Initial Problem-Solving

Turn	Speaker	Spoken Response
1	Teacher	Yes sir, okay.
2	Isaac	Well, so first I didn't like this picture because I thought it wouldn't make more sense to me because I wouldn't know where each line would go.
3	Teacher	Okay, so first you were gonna draw lines and you decided that would be too hard to keep track of.
4	Isaac	Yeah. So then I knew there are 9 people so I just added to take 9 little cupcakes. So then I added 3 [leftover] so then knew I couldn't cut them out so I made a -
5	Teacher	Before you go on to that, tell me this, tell me about why these [9 small circles for the cakes] are marked out again.
6	Isaac	Because since there's 9 people, I counted 'em and once I got to 9 [cakes], I marked out all of those because those had already been aten by the 9 people.

In this exchange, Isaac explained that he tried to use lines to distribute the pieces, but he decided it would not make sense to him (Turn 2), an indication that he was in charge of his decision-making for his strategy. He also explained that he had crossed out 9 of the 12 circles because there were 9 people (Turns 4 and 6).

The teacher and Isaac then established that these crossed out circles meant that every child got "one" so far (not included in the excerpted transcripts here). Next, Isaac explained how he had arrived at the decision to partition the extra cakes into 3 pieces each (see Table 3):

Table 3

Isaac's Explanation about How He Partitioned Extra Cakes

Turn	Speaker	Spoken Response
7	Isaac	So I cut them [the extra cakes] first into twos and I put them together, but I saw that it wasn't working. Next I cutted them into four but it wouldn't work because it would be too big. So I cutted them into 3. I counted...
8	Teacher	How did you know it would be too big?
9	Isaac	Because 4, if you go 4, 8 and then after 8 equals 12. So we'll waste all the cookie and then it wouldn't be enough for everybody.
10	Teacher	Oh, it wouldn't be fair?
11	Isaac	Mm hmm, so I cut 'em into 3 and I counted them so like, 1 for this person, 1 for that person, 1 for this per- so the fastest way that I did it is, I knew there was 3. I just grouped them up.
12	Teacher	Ahh.

In this exchange, Isaac described how he tried partitioning into "twos" (meaning halves) which did not work, into "four" (meaning fourths), and then finally into "three" (see Turns 7 and 11 and part C in Figure 1) When pressed, he also explained how he knew that fours would not work (Turn 9). After deciding that partitioning the extra cakes into 3 pieces each would work, Isaac described how he imagined distributing the pieces 3 at a time ("….I just grouped them up" in Turn 11).

This interaction continues and Isaac has the opportunity to differentiate between how much each person would receive from how much was distributed to everyone. His teacher created space for confusion and laughter as she said much of what Isaac had already shared. By the end of the interaction, Isaac had concluded that the answer to the problem was $1\frac{1}{3}$ cakes, which he represented using invented notation as $\frac{1}{3}$ and $\frac{1}{\text{hol}}$ (see part D in Figure 1).

Isaac: In-Chargeness. Isaac was consistently in charge of his problem-solving and articulation of his problem-solving. He decided what to share, including his guesses at the partitions for the remaining three cakes. He continued to be in charge throughout the interaction.

Isaac: Mathematical Activity. Isaac chose to begin with sharing wholes, and then tried halves and fourths before deciding on "threes." This strategy is an early direct modeling strategy, where he used trial and error to determine which partition would work so that everybody got the same amount. Isaac elaborated on his problem-solving decisions and articulated problem-solving decisions, including how to represent quantities, how to connect quantities, which partition to use, and why he chose a particular partition.

Final Thoughts

Research on whole group instruction and small group work has documented the high quality learning opportunities that occur given instruction that includes rich mathematical tasks and participation structures that support children's participation (Empson & Jacobs, 2021; Webb et al., 2009) and the scenarios teachers find challenging to respond to (Franke et al., 2009; Munson, 2019). My study contributes to this body of research by presenting taking up space as a useful framework to understand children's opportunities to learn during one-on-one classroom interactions, a participation structure often overlooked during classroom instruction. Furthermore, my study introduces a dimensional analysis that defines and measures learning opportunities during instruction more precisely, including an expansive lens that begins to characterize how children reveal more of who they are as a way to engage in sensemaking.

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A STUDY OF AN INTERVENTION PROGRAM: BUILDING ELEMENTARY STUDENTS' COMPUTATIONAL FLUENCY

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This research reports the findings of a mixed-methods study of an intervention program designed to improve elementary students' fluency with math facts. Students participating in the intervention made significant gains in grade level content understanding during the time the intervention was administered. Key features of the intervention included a focus on developing student use of computational strategies and employment of multiple progress monitoring measures.

Fluency with basic mathematics facts is a cornerstone of mathematical literacy. Formally developing students' computational fluency typically occurs in the early elementary grades and extends into the upper elementary levels and middle school. Researchers have found that computational fluency is associated with mathematics achievement in the later years of schooling and is foundational to quantitative reasoning and mathematics achievement (Cason et al., 2019). For students who struggle with mathematics, schools often provide targeted instruction to address concepts and skills students are still developing. This additional Tier-2 instructional support is provided through intervention within a multi-tiered system of support. Systematic instruction, use of visual representations, and consistent progress monitoring are characteristics of strong intervention systems (Bryant et al., 2011).

Fluency with basic mathematics facts, or computational fluency, rests upon processes of efficiency, accuracy, and flexibility with the operations of addition, subtraction, multiplication, and division (Russell, 2000). Efficiency involves employing a strategy through which an answer to a math fact (i.e., $6 + 7 = ?$, $5 \times 3 = ?$) is known or attained with automaticity. Accuracy involves arriving at a correct solution but can also refer to employing appropriate strategies and being precise in recording processes. Flexibility involves drawing upon a repertoire of strategies for solving different problems and being selective about the most appropriate strategy to use for a particular problem (Russell, 2000). At the elementary level, developing fluency with operations of whole numbers is typically a strong focus.

The purpose of this study was to document a tier two mathematics intervention program and its impact. This program was designed for a rural elementary school in the Pacific region with the intent of developing students' fluency with mathematics facts. Ms. Jessica, a veteran teacher-leader at the school, designed and implemented the intervention program with the instructional support of an educational assistant, Amy (both pseudonyms). Specifically for this study, we were interested in the extent to which this intervention focused on building students' fluency with computing mathematics facts supported students with grade level content. The study also describes the design of the intervention and how it functioned to provide supplemental instruction for elementary students in developing computational fluency. Two questions guide this inquiry:

1) What are the key features of this intervention system?
2) During the intervention, are there significant improvements in students' learning of broader grade level concepts and skills?

Methods

A two-phase mixed methods approach was appropriate to address the purpose of the study. Qualitative methods were used to describe the intervention and quantitative methods were used to determine the significance and the effects of the intervention on student success with grade level content.

In the first phase of the study, we examined existing documentation of the intervention and interviewed Ms. Jessica, who designed and led the implementation of the program. The goals for this phase were to describe the structure of the intervention program within the school, the mathematics targeted during the intervention sessions, and the nature of the intervention instructional materials and assessments. Data sources included documentation of timelines, notes on content addressed during each intervention session, sample lessons and student pages from instructional and assessment materials, and researchers' notes from semi-structured interviews. We also gathered and analyzed documentation about the design of the progress monitoring materials and assessment tools utilized throughout the intervention.

In the second phase, we analyzed one progress monitoring measure of de-identified data to quantify changes in students' mathematics abilities over the fall semester. The school used the *i-Ready Diagnostic* (Curriculum Associates, 2011) assessment to track student progress with overall grade level content and to identify students for the intervention. The data included the amount of time the student needed to complete the assessment, raw scores, and scaled scores. Students who were two grade levels behind their peers were identified for the intervention.

Among these students, a few were also receiving special education services. Through interviews with Ms. Jessica, we decided that the combination of support services would create a confounding variable and we therefore decided to exclude data on students who were also receiving special education services. We also excluded data missing a pre or post score. This resulted in sample sizes of $n = 15$ at Grades 2–3, and $n = 75$ at Grades 4–6. We used pre and post data to conduct a paired-samples, two-tailed t-test to determine mean differences in student scores.

Findings

Phase 1: Description of the Intervention Program

Approximately 95 Grade 2–6 students who were assessed to be two grade levels behind participated in the intervention. After one round of progress monitoring, Ms. Jessica made an intentional decision to focus on students who did not have Individualized Education Programs (IEPs) because students who had IEPs did not seem responsive to the intervention. Additionally, students with IEPs had very specific learning needs that were being addressed through another intervention program through the school's special education department. The school uses the mathematics program, *Stepping Stones* (Burnett & Irons, 2022), as its core mathematics curriculum.

The intervention included daily mathematics instruction in addition to the learning experiences students received in their general education classrooms. Students were grouped by grade level with no more than six students in a group. Each instructional session lasted for approximately 15 minutes.

The intervention was implemented by Ms. Jessica, in coordination with Amy, who provided the instruction. Amy did not have additional formal training on the instructional approach employed within the intervention. Implementation was in Ms. Jessica's classroom, where it was possible for her to observe the instruction and have immediate and direct communication with Amy. Ms. Jessica planned the lessons and worked closely with Amy, preparing materials, and providing guidance, support, and direction as needed. In addition, she monitored the enactment of the lessons and would model strategies with different representations as needed. In turn, Amy regularly asked Ms. Jessica questions and for additional support when preparing for instruction. They organized their work and kept track of daily instructional strategies and progress monitoring assessment data on spreadsheets.

The emphasis of instruction was on teaching computational fluency strategies that were aligned with the strategies taught in the main curriculum. Multiple representations were used when students needed them. Ms. Jessica did not believe the use of timed tests supported students in developing computational fluency and there was a focus on strategic thinking rather than on rote memorization. Instruction in the general classroom and the intervention sessions were not coordinated temporally, however, strategies were retaught during the intervention sessions following their introduction in the general education classes. Materials from the *Math Box of Facts* (Burnette & Irons, 2022) were used for instruction in the intervention.

Computational Fluency Strategies

The focus of instruction was on developing students' use of different strategies in addition and subtraction in Grades 2 and 3, and multiplication and division in Grades 4, 5, and 6. During the sessions, students were retaught strategies for computing that aligned with strategies taught in the general education classroom. The sequence of strategies began with less complex and progressed to more complex strategies. A sample list of the strategies that were emphasized during instruction are shown in Table 1.

Table 1

Sample of Computation Strategies Emphasized During the Intervention

Addition Strategies	Multiplication Strategies
Adding 0	Grouping Strategy
Adding to Make 10	Multiplying by Doubling
Adding Doubles	Double Double Strategy
Adding Doubles plus 1	Building Up & Down Strategy

Progress Monitoring

The intervention utilized three progress monitoring measures that provided assessment data of different types of students' mathematical knowledge and skill in computational fluency: 1) The *i-Ready* assessment (Curriculum Associates, 2011); 2) *Math Running Records* (Newton, 2016), and 3) weekly timed tests using materials from different sources. The range of these measures provided an overall assessment of students' mathematical knowledge and skill with grade level content as well as their current skill level in computational fluency. They also helped determine when students were ready for more complex strategies, which was key to the intervention. The *i-Ready* assessment (Curriculum Associates, 2011) was administered three times, once at the beginning, middle, and end of the school year. Students completed this assessment in a self-paced format on a digital device. The assessment contained items on a range of grade-level concepts and skills and were not specific to assessing students' computational fluency in whole number operations.

A second measure was the *Math Running Records* (Newton, 2016), designed to assess students on the dimensions of speed, accuracy, flexibility, and efficiency of basic fact fluency. This assessment provided a sequence of strategies for computing basic facts. Organized by different computational strategies, *Running Records* assessments were administered individually to students approximately every four weeks. Ms. Jessica found these assessments useful because they were comprehensive and gave insight into what strategies students used the most and how to continue their progress. The assessments rely on observations of student use of strategies and are designed to identify an instructional starting point based on where students struggle with a particular strategy. The assessment provided specific information about students' facility with different strategies, which aligned well with the overall emphasis of the intervention.

The third measure was weekly timed tests administered to students individually to determine their automaticity with facts using the strategies they had been learning in that week. A range of materials were used for this assessment, including worksheets from The *Math Box of Facts* (Burnette & Irons, 2022) and *Math Running Records* materials.

Overall, two different dimensions of computational fluency were assessed on a regular basis during the implementation of the intervention. The *Math Running Records* assessed students' use of computational strategies, while the weekly timed tests assessed students' automaticity with computational problems. The *i-Ready* assessment (Curriculum Associates, 2011) provided data on students' knowledge and skill on overall grade level content. Based on daily and weekly progress monitoring data, Ms. Jessica determined the strategies that would be the focus for daily instruction for each student.

Phase 2: Measuring Change in Mean Scores

To investigate the effects of the intervention on students' attainment of general grade-level concepts and skills during this same period, de-identified progress monitoring raw scores from two administrations of the *i-Ready* assessment (Curriculum Associates, 2011) were used. Raw scores were used rather than scaled scores because the raw scores allowed for greater variability in the data. The assessments were administered in accordance with the guidelines provided by the publisher. There were no concerns with regard to the validity of the assessment process. Students in the intervention program were assessed at the start of the school year in August 2021, and again in December 2021. Scores on these assessments served as pre and post data in our analysis.

The mean pretest score for students in the Grade 2–3 group was M = 383.8, SD = 29.45, and a mean posttest score was M = 402.13, SD = 32.49. To further investigate the apparent increase in scores, we conducted a paired-samples *t*-test for the matched pre-post data. After excluding three students due to missing a pre/post pairing, the results showed a statistically significant gain between the pre and post scales ($t(14) = -4.406$; $n = 15$; $p < .000$).

The data were further analyzed to determine an effect size. We found a moderate effect size of the intervention ($d = .59$, 95% CI [9.41, 27.26]), per benchmarks suggested by Cohen (1988). See Table 2 for results.

Table 2

Change in Scores from August to December, for the Grades 2–3 Intervention

						95% Confidence Interval of the Difference				
	Mean	Std. Dev	Mean	Std. Dev	Std. Error Mean	Lower	Upper	*t*	df	Sig. (2-tailed)
Pre	383.8	29.45	18.33	16.11	4.16	9.41	27.257	-4.406	14	.000
Post	402.13	32.49								

The same analysis was conducted with data from the Grade 4–6 group. Data from two students were excluded due to missing a pre/post pairing. The mean pretest score for this sample was M = 437.56, SD = 30.35, and the mean posttest score was M = 457.03, SD = 30.28. The results from a paired-samples *t*-test showed a statistically significant gain $t(74) = -11.62$, $n = 74$, $p < .000$. We then checked for a statistically significant effect and again found a moderate effect size of $d = .642$, with a 95% CI [16.13, 22.8]. This finding provides further evidence that the intervention improves students' procedural fluency on grade level content. See Table 3 for results.

Table 3

Change in Scores from August to December, for the Grades 4–6 Intervention

						95% Confidence Interval of the Difference				
	Mean	Std. Dev	Mean	Std. Dev	Std. Error Mean	Lower	Upper	*t*	df	Sig. (2-tailed)
Pre	437.56	30.35	19.47	14.51	1.68	16.13	22.8	-11.62	74	.000
Post	457.03	30.28								

Discussion and Implications

Several key features of the intervention system were central to how it functioned and highlighted systematic instruction and assessment. Consistent communication and collaboration between Ms. Jessica and Amy were effective in providing high quality instruction that targeted students' needs. Ms. Jessica's words, "you really need a good instructor" highlight the importance of this aspect of the system. Ms. Jessica and Amy regularly communicated when students applied a strategy effectively and experienced challenges. This direct communication led to their ability to systematically enact instruction, adjusting instructional strategies and use of representations when students needed them. In addition, Ms. Jessica and Amy documented each student's daily, weekly, and quarterly progress.

Another key feature was the coordination of components of the instruction that led to a strongly aligned and focused intervention system. Instruction on computational strategies clearly aligned with progress monitoring assessments. The strategies students had previously worked with and were currently working on in the general education classroom were coordinated with the instruction in the intervention.

Ms. Jessica's robust pedagogical content knowledge for developing computational fluency in students was another key feature. Her decision to focus on developing students' flexibility with different computational strategies and providing instructional support to Amy for teaching and use of representations aligns with what is supported in research. Ms. Jessica was clear about limiting timed tests to assessing student automaticity and thus they were not used for instruction. Documentation of the intervention sessions indicated a very clear focus on using strategies to compute different facts. Furthermore, the strategies followed a progression toward higher complexity of strategic thinking and development of flexibility in employing different strategies. It was evident that there was a constant effort to push students toward utilizing more complex strategies once they had mastered earlier ones.

The last key feature was the employment of multiple progress monitoring measures to document student growth and assess different dimensions of computational fluency. *Math Running Records* (Newton, 2016) assessed students' accuracy and flexibility with computation strategies, and weekly timed tests assessed students' automaticity with computation facts. These measures were used as formative assessment to guide instruction during the intervention sessions. *i-Ready* assessed students' overall mathematics knowledge and skill on grade level content. This measure provided data that were used to determine the impact on grade level mathematics. Our analysis showed that students made significant gains in their grade level content understanding during the time the intervention was administered. Sixty percent of the Grade 2–3 students in the study and approximately 78% of the Grade 4–6 students increased by at least one grade level in content knowledge. Thus, although the intervention required students to miss their regular classroom instruction, the overall effect appeared to support success with (re)learning computational strategies.

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DIFFICULTY ASSESSING ELEMENTARY STUDENTS' MULTIPLICATION FACT FLUENCY

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This paper examines an initial written assessment of third- and fourth-grade students' multiplication fact fluency levels. The assessment operationalized fluency as efficient, accurate, and flexible. The first iteration was given to six students in third- and fourth- grade and was scored using a rubric with levels of beginning, developing, emerging, and accomplished. Students' reasoning strategies were also noted. Results indicated that the reasoning strategies used by the students spanned all phases of the development of fluency. Difficulties assessing fluency were noted and in future iterations the student's strategy selection will be taken into account on the rubric. Another difficulty was a lack of measuring a student's flexibility when solving the problems. Therefore, in future iterations worked examples will be included.

The calls to increase students' fluency with multiplication facts abound, yet the field lacks a clear consensus on two items critical to this call. First, what is mathematical fluency? And second, how can we assess fluency with multiplication facts? Previous research has assessed automaticity or memorization of multiplication facts using written assessments that are typically timed (e.g., Burns et al., 2019), but these assessments do not attempt to incorporate aspects of fluency that are a focus of NCTM's (2014) definition, such as flexibility in reasoning strategies. In contrast, Bay-Williams and SanGiovanni (2021) have developed an interview protocol and scoring rubric for assessing multiplication fact fluency. This assessment operationalizes fluency as efficiency, accuracy, and flexibility, but could not reasonably be scaled in many classroom environments. Therefore, the overarching goal of this study is to create a written assessment of multiplication fact fluency that is consistent with NCTM's (2014) and Bay-Williams and SanGiovanni's (2021) definitions of fluency yet can be administered to classrooms. Here, we report on initial attempts at generating such an assessment as well as difficulties encountered, and ask, what does a written assessment meant to measure multiplication fluency among third- and fourth-grade students, operationalized as accuracy, efficiency, and flexibility, look like? We also ask, what strategies do third- and fourth-grade students demonstrate to solve multiplication problems on a written assessment?

Literature Review

NCTM (2014) defines the components of procedural fluency as efficiency, flexibility, and accuracy. *Efficiency* means that students can solve "a procedure in a reasonable amount of time by selecting an appropriate strategy" and that efficiency is when students select reasonable strategies to solve problems, solve problems relatively quickly, and change strategies when it is beneficial to do so (Bay-Williams & SanGiovanni, 2021, p. 3). *Flexibility* means “knowing multiple procedures and applying or adapting strategies to solve procedural problems” (Baroody & Dowker, 2003, as cited by Bay-Williams & SanGiovanni, 2021, p. 3), and flexibility can be observed in students’ problem solving if they change or adapt strategies when it is beneficial to do so or apply strategies to new problem types (Bay-Williams & SanGiovanni, 2021). Finally, *accuracy* implies completing a procedure correctly, which can be observed by generating correct steps to a procedure and identifying a correct answer (Bay-Williams & SanGiovanni, 2021).

Despite the comprehensive nature of this definition of fluency, many researchers continue to assess fluency through timed tests (Burns et al., 2019; Burns et al., 2006; Codding et al., 2011) and view fluency as automaticity or memorization (Poncy et al., 2010). However, a view of fluency as defined by NCTM (2014) asserts that memorization of math facts will develop *as a result* of fluency, and cannot, therefore be used as a measure of fluency (Baroody, 2006).

Baroody (2006) describes the development of fluency to progress through three phases. Phase 1 involves the use of counting strategies. For instance, on a multiplication problem, a student in phase 1 might skip count, or draw objects to represent a multiplication problem and count them one by one. Phase 2 of fluency is the use of reasoning strategies, which is described as “using known information to determine an unknown combination logically” (Van de Walle et al., 2019, p. 190). For instance, a student who has not memorized the fact 6×7 might know that 3×7 is 21, and double 21 to determine that 6×7 is 42. Although this strategy does not demonstrate memorization of 6×7=42, it does demonstrate fluency because the student selected an efficient strategy and applied it accurately. Phase 3 is mastery, or automaticity, at which point students are likely to say that they “just know” the answers to multiplication facts. Bay-Williams and Kling (2019) add that mastery is most likely to occur first with foundational multiplication facts – 0s, 1s, 2s, 5s, 10s, and perfect squares – and later include the 3s, 4s, 6s, 7s, 8s, and 9s. The 7s and 8s are typically mastered last because they are more reliant on break apart reasoning strategies (Bay-Williams & Kling, 2019). Thus, the use of timed multiplication tests may identify students who have reached the mastery phase of multiplication fluency but provides no information as to whether students have memorized (and will subsequently forget) those facts or whether they have developed to the point of phase 3. Therefore, in this study, we describe initial efforts to develop a written instrument to measure procedural fluency with multiplication facts among third- and fourth-grade students, and report on the types of strategies that the students used.

Methods

Existing Clinical Interview Instrument and Rubric

The initial instrument was based on Bay-Williams & SanGiovanni's (2021) clinical interview protocol and rubric, which includes six interview questions, and the format asks students to solve each problem and to explain how they know their answer is reasonable. The interviewer asks students if there are other ways to solve and how they decided which one to use. The corresponding rubric categorizes students' fluency with multiplication facts as beginning, developing, emerging, or accomplished (Table 1).

Table 1

Example of the Fluency Rubric (adapted from Bay-Williams & SanGiovanni, 2021, p. 163)

Category – Definition of the category • Example student solutions on the task 6x4 • *Explanation*
Beginning (1) – "Knows one algorithm or strategy but continues to get stuck or make errors." • *6×4 is 23. I counted up to 6, four times. I don't know another way.*
Developing (2) – "Demonstrates efficiency and accuracy with at least one strategy/algorithm, but does not stop to think if there is a more efficient possibility." • *6×4 is 24. I skip counted by 6. There might be another way, but that's the way I always do it.*
Emerging (3) – "Demonstrates efficiency and accuracy with several strategies, and sometimes selects an efficient strategy, though still figuring out when to use and not use a strategy." • *6×4 is 24 I knew 6x5 is 30, so then I counted back 6 to get 24. I guess I could have done 5×4, though, and then added 4 more.*
Accomplished (4) – "Demonstrates efficiency and accuracy with several strategies and is adept at matching problems with efficient strategies…" • *6×4 is 24. I know 6×2=12, so I doubled that. I could have done 5×4=20, then added 4. Or, I could skip count or add 6 four times. I just thought doubling 12 would be easiest.*

Initial Instrument and Scoring

The first iteration of the written instrument utilized six multiplication problems in the following order: 6×2, 4×9, 6×7, 8×3, 6×4, and 22×5. 6×2 was selected as the first problem because (1) we anticipated that most students who had been exposed to instruction on multiplication could solve this problem with at least one strategy, and (2) we anticipated that 6×2 might prime students' thinking to double 6×2 to calculate 6×4. 4×9 was selected because it can be generated efficiently through at least two strategies using foundational facts: (1) doubling 2×9 (a foundational fact), or (2) a compensation strategy of 5×9 (a foundational fact) minus 9 or 4×10 minus 4. We selected 6×7 and 8×3 because the sevens and eights multiplication facts are typically the last to be memorized (Bay-Williams & SanGiovanni, 2021), so we hoped to elicit strategy use, even if students had memorized the first three facts. 22×5 was selected to test students' strategies with a two-digit by one-digit multiplication problem to determine (1) if they would use a standard algorithm, (2) if they would utilize the foundational fact 5×2 to calculate 5×20, and (3) if they would extend efficient strategies from one-digit multiplication facts to a situation of two-digit multiplication.

After writing the answer to each problem, students were asked the same three questions:

1. How did you solve the problem? *You may draw a picture if that is helpful.
2. If you *just knew*, how would you explain how to solve the problem to a friend that did not have this problem memorized? *You may draw a picture if that is helpful.
3. Do you know more than one strategy that could be used to solve this problem? *You may draw a picture if that is helpful.

These questions were meant to elicit the students' strategy use on the problem and to gauge whether they could identify a second strategy.

Students' fluency on the initial written assessment was scored using the rubric in Table 1. For each problem, we noted whether the answer was correct or incorrect, what strategies they used, how many strategies they used, and attributed a score from 1 (beginning) to 4 (accomplished), based on their written explanations. Then, considering the assessment as a whole, we calculated each student's mean score; means with a decimal greater than or equal to 0.5 were rounded up, and with a decimal less than 0.5 were rounded down. All strategies were coded across the assessment, the total number of strategies was counted, and an overall fluency category was assigned based on the mean fluency score (e.g., mean score = 3, overall category = emerging).

We anticipated that on the written assessment there might be some students who had a lower mean score because they only used one strategy on each problem, but whose work across the assessment indicated a higher fluency category because they used a diversity of efficient strategies across the assessment (e.g., mean score = 2, but number and type of strategy codes could imply overall category = emerging (3)). In these situations, we considered the written assessment holistically using the rubric in Table 1, rather than relying only on the mean score.

Data Collection and Analysis

Approval for this study was given the IRB at Oklahoma State University (IRB-23-263). We collected data from six students in a rural school district in the midwestern U.S. Two third-grade students and four fourth-grade students participated. The students were selected for convenience. All parents gave permission for their child to participate, and each child gave assent. Each student was told to answer each question as best they could and to not erase any of their work or skip any questions. If they did not know the answer they were to write "I don't know." Students were not timed.

Results of the Initial Written Assessment

The results of the initial written assessment are shown in Table 2. Table 3 shows the strategies utilized by each student, and their overall fluency category. The strategies used by each student in Table 3 are grouped by fluency level: counting, reasoning strategies, and algorithms.

Table 2

Initial Written Assessment Results by Item and Overall

Student	Grade	6×2	4×9	6×7	8×3	6×4	22×5	Mean	Category
1	3	2	2	2	2	2	2	2	Developing
4	3	2	3	1	2	2	2	2	Developing
2	4	2	3	2	3	3	3	3	Emerging
3	4	4	3	2	3	2	2	3	Emerging
5	4	2	2	3	3	2	1	2	Emerging
6	4	2	2	1	2	2	2	2	Developing

The counting code indicates that the child's strategy was to count by ones or skip count. For instance, student 2 wrote, "count to six two tims [*sic*]" in response to 6×2. Reasoning strategies included adding (i.e., repeated addition), or adding combinations which was indicated by adding at least two combinations of multiples of one factor. For example, to multiply 22×5, Student 1 wrote, "44+44+22=105." The student presumably doubled 22 to find 44, rather than adding 22 five times. Visual reasoning strategies included number lines and hundreds of charts, although both students described using these visual representations without drawing them. Finally, student 5 used a compensation strategy to solve 4×9 and 6×7. For instance, they wrote, "5×7=35+7=42."

Table 3

Strategies Used by Students

Student	Grade	Strategy Codes			Total	Category
		Counting (1)	Reasoning (2)	Algorithms (3)		
1	3	Count	Add, Add combinations	--	3	Developing
4	3	--	Add, Add combinations, Additive picture	--	4	Developing
2	4	Count, Skip count	Add	Standard algorithm	4	Emerging
3	4	Count	Add, Add combinations, Number line	Partial products, Standard algorithm	6	Emerging
5	4	--	Add, Add combinations, Hundreds chart, Compensation	--	4	Emerging
6	4	Count, Skip count	Add	--	3	Developing

The two third-grade students both were determined to have developing fluency overall (Table 2). These students tended to use counting strategies or additive strategies to solve the problems or to explain the problem to a friend. The third-grade students also tended to only have one strategy to solve each problem, which is the primary reason that they were both attributed the developing category for multiplication fluency. The exception is that on the problem 4×9, student 4 used two strategies. They described that they solved the problem by adding 18+18, but to explain the problem to a friend, drew a picture that included 4 groups of 9 squares, with a plus sign in between each group. This suggests that the student was aware that adding their doubles facts (18+18) was more efficient than drawing a picture and adding the four 9s, which is a characteristic of emerging fluency (level 3). When taking all six problems into consideration, however, student 4's typical reasoning was limited to one strategy on each problem, and so the developing category was attributed for their overall multiplication fluency.

In comparison, three of the four fourth-grade students were attributed emerging fluency and one had developing fluency with multiplication facts. The fourth-grade students' strategies also included counting and additive strategies but extended to other visual strategies (number lines and hundreds charts), skip counting, compensation, partial products, and the standard algorithm. Student 5's overall reasoning level was determined to be emerging, despite a mean of 2.17. The researchers made the decision to indicate this student's level of fluency was emerging, rather than developing, because the student demonstrated accuracy with several strategies (repeated addition, compensation, hundreds chart, and adding combinations) across the assessment, and also from the researchers' perspectives, the student used these strategies in situations that were efficient. For instance, the student suggested repeated addition (6+6) to solve 6×2 but to solve 6×7 they used compensation (5×7=35+7=42). Although the student did not explicitly state that they used compensation because it was more efficient, the consistency with which the student selected what we perceived to be an efficient strategy suggests at least an implicit awareness of the appropriateness of different strategies for different problems.

In combination, the strategies that were used by these third and fourth grade students spanned all three phases in the development of fluency: counting, reasoning strategies, and algorithms. None of the students with developing fluency reasoned with algorithms, and they averaged 3.33 strategies each. Two of the students with emerging fluency demonstrated algorithms in their reasoning, and the students with emerging fluency averaged 4.67 strategies each.

Difficulties Observed in Assessing Fluency

Based on this first iteration of an assessment meant to measure third- and fourth-grade students' multiplication fluency, there are several key takeaways. First, the averaging of item scores across the assessment was consistent with the researchers' holistic assessment of the students' reasoning for five out of six students. Because of the small sample, it is impossible to tell whether student 5 (whose mean score was 2 but whose overall fluency level was considered emerging) is an anomaly. Thus, moving forward, we will develop a scoring system that is more holistic and takes into account not only the student's rubric score on each individual item but also the types of strategies that each student uses, and whether those strategies demonstrate counting, reasoning, or algorithms.

A second difficulty illustrated by the results of the initial instrument allowed us to measure students' accuracy with multiplication facts and their selection and use of efficient strategies. We also intended to measure students' flexibility in reasoning by assessing the number of strategies that the students' demonstrated across the assessment, however this analysis proved more consistent with students' selection of efficient strategies than it did with their flexibility in strategy selection. Thus, in a future iteration, we intend to include questions that are formatted as worked examples. On these, students will be shown a correct student solution demonstrating a strategy, such as compensation or doubling, then students will be asked if they can use the same strategy to solve a different problem.

Conclusion

Although the field has made strides in defining multiplication fluency as more than the memorization of facts (NCTM, 2014), it remains difficult to assess multiplication fluency at scale when it is operationalized in such a complex manner. This initial assessment shows that aspects of fluency can be assessed among third- and fourth-grade students with a written assessment, although there is much work that remains to be done.

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EXPLORING MOTIVATION AND MATH APPS: A THIRD GRADER'S STORY

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Math app use has continued to become more prevalent in K-12, with most research examining students' achievement. The present study is part of a larger exploratory multiple-case study examining students' mathematical motivation as it relates to math apps. I utilized the basic needs from self-determination theory and the interest task value from expectancy-value theory to understand motivation. This paper introduces the case of Sarah, a third grader whose motivation to use math apps is primarily determined by autonomy and interest.

Over the last few decades, national and worldwide use of blended learning (BL) has escalated (Barbour, 2018). While blended learning (BL) has become an umbrella term to describe programs that utilize online learning (Hrastinski, 2019), math apps are a common technology used in many BL programs (Cleveland-Innes, 2018). These apps have gained significant traction worldwide with math apps such as DreamBox, Zearn, and IXL becoming widely utilized. In 2022, a national survey of teachers in the U.S. found that over a third of the additional instructional materials teachers used to teach math were math apps (Doan et al., 2022). While most research on math apps has focused on learning outcomes and achievement (Griffith et al., 2020), very little research, with Ke (2008) as a notable exception, has looked at motivation related to math apps where the math apps are accessible through a laptop. Because laptops are the most common device in elementary schools (Gray & Lewis, 2021), I believe focusing on math apps that can be played on laptops is important. Existing literature indicates that motivation is an important aspect of students' learning experiences (Middleton et al., 2017) with it being recognized as an important mediator of mathematics learning and achievement (Schukajlow et al., 2023). Given the significant role motivation plays in students' academic well-being and recent increases in math app use in K-12, the research question that guided this study was: What is the relationship between math apps and students' motivation in mathematics?

Literature Review and Framing

Motivation is generally viewed as the process of initiating and sustaining behavior (Schunk et al., 2014), or why we choose to engage in and persist with an activity. In the context of school mathematics, mathematical motivation is a reason for engaging in mathematics, that is, a student's reasons for "what they choose to do, with whom, and to what ends" (Middleton et al., 2017, p. 675). The focus of my study is to examine the relationship between math apps and students' mathematical motivation. Examining a small body of motivation research related to elementary school math app use, Ke (2008) found that fifth-grade students utilizing a math app that emphasized fact fluency and "skill-and-drill" problems, Astra Eagle, experienced a significant positive change in motivation and attitude toward mathematics. The study utilized a "Teams-Game-Tournament" cooperative structure where teams of students battled against other teams on the math app. Ke (2008) posits that "cooperative goal structure, with performance-contingent rewards, encourages interpersonal association and sense of relatedness" (p. 441) which led students to experience a significant positive change in motivation and attitude toward mathematics. Given technology is a key part of students' learning environment and alters how students relate to mathematics (Borba et al., 2016), I view motivation as situated.

Considering motivation from a situated perspective requires understanding motivation as a product of being, learning, and interacting within one's environment (Ryan & Deci, 2000). This view of motivation is consistent with self-determination theory (SDT) which I use to guide my operationalization of motivation. SDT rests on the understanding that all human beings have the same three basic psychological needs: the need for competence, social relatedness, and autonomy (Ryan & Deci, 2000). Social relatedness is defined as "a sense of affiliation with or belonging to others to whom they would like to feel connected" (Cook & Artino, 2016, p. 999) and is grounded in the view that motivation is a social activity and context dependent. Autonomy "refers to the opportunity to control one's actions" (Cook & Artino, 2016, p. 999), and "competence refers to the perceived ability to master and achieve" (Cook & Artino, 2016). One of the reasons why I operationalize motivation using SDT is because of the unique opportunities math apps provide for autonomy, social relatedness, and competence (see Table 1).

Additionally, I utilize the interest task value from expectancy-value theory (EVT) (Eccles et al., 1983) as research has shown students enjoy playing math games and math apps (Shin et al., 2012) and results from my pilot study indicated this is an essential component of students' motivation missing from my operationalization of SDT. Eccles et al. (1983) define interest as "the inherent, immediate enjoyment one gets from engaging in an activity" (p. 89). As part of my theoretical framing, I theorize how features of math apps may interact with the basic needs outlined by SDT and the interest value from EVT in Table 1.

Table 1

Features of math apps and their theorized interaction with motivation.

Components of motivation	Features of BL and self-directed math apps						
	Ability to play multiplayer	*Multiple game modes*	*Immediate feedback*	*Reward Mechanism*	*Captivating graphics*	*Emergent Narratives*	*Dynamic/ adjustable*
Autonomy	X	X		X		X	
Social relatedness	X						
Competence			X	X			
Interest	X	X	X	X	X	X	X

Background and Methods

This study was part of a larger exploratory multiple-case study that examined third graders' motivation and math identity related to their weekly use of math apps. The multiple-case study consisted of eight cases that were selected from a classroom of participants based on their different mathematics identities and motivations. I focused on Sarah's case for this preliminary study because she was revelatory (Yin, 2016) about her thoughts regarding the math apps she utilized. Particularly, Sarah clearly articulated how the different utilized math apps related to her autonomy, competence, social relatedness, and interest while other third graders had a harder time expressing their motivation as it related to math app use. The two math apps utilized every week in Sarah's class were Reflex and Prodigy. At the time of the study, Sarah's classroom was learning about different strategies to solve multidigit multiple and division problems. Sarah's classroom is a part of X Elementary School, a large public elementary school located in the southern United States. X Elementary School has an economically disadvantaged student enrollment of 16% and over 70% of the school's students scored at or above the proficient level on the state math test.

Drawing on the three basic needs outlined by self-determination theory (Ryan & Deci, 2000) and the interest task value from expectancy-value theory (Eccles et al., 1983), I used autonomy, competence, relatedness, and interest as a priori codes (Miles et al., 2014) to analyze Sarah's responses to an eighty-minute semi-structured interview protocol (Rubin & Rubin, 2011), a thirty-minute follow-up interview, weekly and pre- and post-surveys (Likert where 1 = Completely Not True & 5 = Completely True), and field notes from weekly observations. I utilized analogies to explain each component of motivation to help Sarah conceptualize and effectively engage with interview questions and survey items. Using autonomy, competence, relatedness, and interest to serve as column heads of the motivation profile I created for Sarah, I analyzed her interviews, survey responses, and my field notes from weekly observations by creating a meta-matrix (Miles & Huberman, 1994) to keep track of her math motivation separated from technology (math motivation) and related to technology (math technology motivation).

Findings

I start by summarizing Sarah's mathematical motivation and mathematical technology motivation (see Table 2). Overall, Sarah expressed an enjoyment and interest in math that came from being successful. Several features of the math apps Reflex and Prodigy afforded varying opportunities for autonomy, competence, relatedness, and interest. In the sections that follow, I further explore Sarah's mathematical motivation and her motivation related to math apps.

Table 2

Sarah's Math Motivation Profiles

Sarah's Math Motivation				
Autonomy	**Competence**	**Relatedness**	**Interest**	
Autonomy wasn't a major factor in determining Sarah's motivation. She didn't mind doing assigned work and felt that assigned problems were usually harder than the ones she'd choose on her own.	While Sarah didn't view herself as the best at math, she saw herself as good at math and believed she could solve any type of problem.	Sarah enjoyed interacting with her classmates. She noted that being able to see people's expressions was important in feeling connected with others.	Sarah found math intrinsically fun and enjoyed the feeling of achievement. Most of her joy and interest in math was rooted in earning good grades and achieving academic success.	
Sarah's Math Technology Motivation				
Autonomy	**Competence**	**Relatedness**	**Interest**	**Math App**
Sarah enjoyed being able to choose from many games in Reflex. While there is little choice in the math content on Reflex, this didn't bother Sarah.	Sarah found the math on Reflex to be an appropriate fit and it helped her learn and practice math well.	Since Reflex is done completely independently, there is no opportunity for relatedness. This was not a deterring feature of Reflex for Sarah.	Sarah found the game on Reflex fun and engaging. She also liked the noises and animations on Reflex.	Reflex
Sarah found the ability to move around freely in the game world and having freedom over purchasing items with money to be motivating autonomous features of Prodigy.	Competence was not a motivating factor when deciding to play Prodigy. Sarah noted the math on Prodigy was hard.	Prodigy's feature of batting with and against friends made the app more motivating and allowed for relatedness not found in any other math app.	Sarah enjoyed being able to collect different pets, battle people, and run freely in the game world. These features motivated Sarah.	Prodigy

Sarah's Mathematical Motivation

Sarah expressed in several instances that autonomy was not a major factor in determining her motivation. During a daily fifty-minute free time where students were given a "menu" of items to choose from, Sarah never chose to do paper-and-pencil math. Additionally, her response of neutral to the survey item "I feel free to choose which math activities I do" suggests Sarah perceived math to not fully allow for autonomy. Sarah also indicated that the math done on paper afforded her the option to do math however she wanted. Sarah said, "You can solve it however way you want [on paper]. You can, you could, some people might want to write a question this way, but other people might want to write it this way." Her resulting attitude was an indifference toward assigned and dictated math activities and a lack of autonomy in math.

Sarah believed she was good at math but was quick to note, "I'm not the best person at math." Despite this, she felt no math was too challenging for her. Sarah said, "I get a lot of the problems right and I, and I know how to solve, how to solve them if there, if it's a word problem, or if it says how many more or something." She also responded with a 1 (Completely Not True) to the survey item "I often have doubts about whether I'm good at math" indicating Sarah felt competent about her math abilities. Overall, competence and Sarah's ability to do well on a math activity or problem had little significance in determining if she would do the activity or problem.

While Sarah did not have much to say about relatedness, she did express an enjoyment for group activities and working with her classmates. She articulated that math free from technology allowed her to interact more with classmates and better understand their emotions. Sarah said that doing math on apps "gives you a little bit less [interaction time] than actually talking to each other." Feeling connected to classmates and being able to see their expressions, and thus understand their emotions, was important and motivating for Sarah.

Sarah expressed an interest and intrinsic enjoyment that came from doing math. Her enjoyment was primarily related to achieving success and positive outcomes in math. She said, "I like, um, the feeling when I finish, um, a math question or get it correct." Sarah mentioned that as she got older, she started to care more about her grades and found more enjoyment in doing well. With multiplication, addition, and subtraction, Sarah said, "All the problems are perfectly memorized." She also responded in agreement (response of 4) to the survey item "Math is exciting to me" indicating an enjoyment for math. Overall, Sarah's enjoyment was tied to performing well and achieving good grades and this motivated her to do well in math.

Sarah's Mathematical Technology Motivation

Sarah indicated two features of math apps that allowed for autonomy and made playing math apps enjoyable. First, she expressed that having different choices within a game was a feature that gave her more freedom over her learning. For instance, Sarah said that frequently the apps would allow her to "like open new games on it." The second feature was specific to Prodigy. In Prodigy, she was able to freely move around in the game world, interact with characters of her choice, buy characters and add-ons with money from the game, and feel like "I'm an adult, I can do whatever I want." When she had free time, Sarah would choose to play Prodigy over Reflex and paper-and-pencil math. In summary, both Reflex and Prodigy had features that gave Sarah autonomy, and these game features were important to her.

Competence wasn't a major determining factor in determining Sarah's motivation to practice math on an app. However, she communicated several features about the cognitive demand of math on each app. With Reflex, Sarah said, "Reflex has a bunch of different stages. So it fits how you, what you're learning, and how fast you learn." She also mentioned that if it was too easy or too hard, her teacher could move her up or down a level to ensure she was being appropriately challenged. On a weekly survey given to Sarah after her free time when she chose to play on a math app, she was asked, "Did you feel successful doing the math in each app today?" to which she replied, "Yes, I got most of them right." This indicates that even though competence didn't impact if she chose to play the math app, the math app did affect her feeling of competence. Overall, Sarah's confidence in her mathematical ability made competence a small factor in determining her motivation to engage in a math app.

Sarah only experienced relatedness in Prodigy as this app had a multiplayer feature that allowed her to battle classmates and other users on the app. However, while this was a feature she enjoyed, Sarah pointed out there were aspects of relatedness lost with math apps. For example, when she played on Prodigy, "like some of the avatars, they're just frowny face the whole time and then like mine's just smart, just has the same expression." This made it hard to understand how other people were feeling and gauge classmates' emotions. Sarah also said that math apps didn't allow for as much interaction time as math done without technology. Overall, relatedness was not a significant feature of Prodigy for Sarah.

Interest was the most important factor in determining Sarah's motivation and reason for utilizing a math app. With Prodigy, she said, "I just, I like running, I like the battles and collecting pets and that's really cool." Sarah made an analogy saying, "It's [Prodigy] really fun, it's like a video game, but it's a lot more learning and you don't have to use a little controller." Reflex also had captivating graphics and animations that made the math app enjoyable. On a weekly survey given to Sarah after her free time when she chose to play on a math app, she was asked, "Did you enjoy using each app today?" to which she responded, "Yes, I did not have to do the picture puzzle today." This game detracted from Sarah's enjoyment and illustrated how her motivation decreased when she was forced to play games she didn't enjoy. Overall, the gamified nature of doing math made playing on Reflex and Prodigy fun and motivating for Sarah.

Discussion and Conclusion

While these results are preliminary, they suggest certain features of math apps afford different opportunities for autonomy, competence, relatedness, and interest. For Sarah, interest and autonomy were the largest factors in determining her motivation to engage in a math app. This is supported by research showing most elementary students find math apps fun and interesting (Kay, 2020). Sarah found mathematics void of technology was also fun, but this enjoyment and motivation to do math was primarily tied to performance whereas math on math apps was fun because of the captivating graphics, game choices, and characters. While autonomy was a major factor in Sarah's motivation to play on the math apps, this factor was of little importance when doing paper-and-pencil mathematics. Contrary to Ke (2008), the skill-and-drill math app, Reflex, did not seem to significantly increase Sarah's motivation to do math.

Findings from this study also suggest that different features of math apps that provide unique opportunities for motivation are dependent on the student. One limitation of this study is that I focused on the motivation of one student, which did not allow for a comparison of motivation across multiple participants. Future work could perhaps examine more participants (i.e., more elementary students) or more students across different grades. I consider research on elementary students' motivation as it relates to math app use an important, yet under-researched, area of K-12 mathematics education. With this type of technology becoming increasingly prevalent, it's imperative we continue to investigate dimensions of students' motivation.

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TEACHING ANGLES DYNAMICALLY THROUGH QUANTITATIVE REASONING

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This paper examines how teaching third-grade students to explore dynamic angles through quantitative reasoning helped them construct their robust understanding of angles that reflects multiple angle conceptions. The findings of this study show how tasks bridge the geometric, multiplicative, and non-static natures of angles into a unified construct. The outcomes of this study offer valuable insights into designing effective instructional tasks that can reinforce students' comprehensive understanding of angles through the quantitative reasoning lens.

For more than 30 years, research has generated various discussions about students' conceptions of angles. Studies show that students develop different angle conceptions about angles, such as angles as unions of rays (Clements & Battista, 1989), as wedges (Browning et al., 2007), and as rotations (Confrey et al., 2012). These different conceptions are likely developed through fragmented teaching of angles. The Common Core State Standards for Mathematics (CCSSM) (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSS], 2010) reflect that students learn angles incrementally throughout various grade levels. In kindergarten, students examine angles as corners of a shape. In fourth grade, they learn about the measurement of angles. In eighth grade, students work with angles as rotations.

Despite its central role in mathematics learning, teaching angles in fragmented ways potentially contributed to students exhibiting alternative conceptions about angles that are mathematically problematic. For example, when students only perceive angles as unions of rays, they struggle to identify the specific aspect of an angle to be measured; thus, students generalize that angle measures depend on its side lengths (Keiser, 2004). Conversely, students viewing angles as wedges associate the size of the angle with the area of a sector represented by a wedge (Browning et al., 2007). Confrey et al. (2012) also discussed that students connect their conception of a turn with the fraction of a circle, and then with the degree measure. Even with conceiving angles as rotations, quantifying continuous and multiple rotations is challenging for students (Kaur & Sinclair, 2012). Teaching angles should consider multiple conceptions to foster meaningful comprehension (Freudenthal, 1973). Thus, integrating multiple angle conceptions that show the geometric (union of rays), multiplicative (wedges), and non-static (rotations) natures of angles may be crucial for a deep understanding of the concept and to avoid developing alternative conceptions that are detrimental to mathematics learning. The geometric, multiplicative, and non-static features are the core of dynamic angles.

Theoretical Framework

The instruction of dynamic angles aims to offer opportunities for students to conceive multiple angle conceptions and to illustrate the interplay between the geometric, multiplicative, and non-static natures of angles through quantitative reasoning. Quantitative reasoning is a system of mental operations involving the conceiving of a situation, constructing quantities from the conceived situation, and developing relationships between quantities (Thompson, 2011). Cultivating quantitative reasoning beginning in the early years of schooling may foster adaptable and generalizable reasoning skills, which can further develop in higher education (Smith & Thompson, 2007). Therefore, through quantitative reasoning, this study explores how the geometric, multiplicative, and non-static aspects of angles can be unified into a single construct, shaping the design of mathematical activities for dynamic angles.

In terms of quantitative reasoning for angles, one may conceive the geometric and non-static features in generating angles by rotating a ray or both rays and identify the visualized quantities during the generation process, and then construct the multiplicative relationships between these quantities. Figure 1 shows the outline of the quantitative reasoning framework for teaching dynamic angles and the orchestrating questions in eliciting the three quantitative reasoning processes (Germia, 2022). Each process has corresponding hypothetical qualitative forms of quantitative reasoning about dynamic angles. To shape the teaching of angles dynamically through quantitative reasoning, digital technologies have a significant role in engaging students in generating angles via rotation, tracing the rotation process, and quantifying such a rotation. Following these features of digital technologies, the researcher designed instructional tasks in GeoGebra, a dynamic geometry environment, to elicit students' quantitative reasoning about dynamic angles.

Figure 1

Quantitative Reasoning Framework for Teaching Dynamic Angles (Germia, 2022)

Quantitative Reasoning Processes	Student Reasoning
Conceiving the angle situation: What did you create? What did you notice?	***Bridging the three angle conceptions*** • Given an angle situation, students may conceive angles as a union of two angle sides, as a rotation, and as a wedge.
Identifying quantities: What is changing? How do you make the angle bigger or smaller?	***Reasoning about quantities*** • Identifying quantities that are changing and not changing. • Construct smooth and chunky images of change.

Constructing relationships between quantities: How is it changing? How are the changes related to each other?	***Comparing Quantities*** • Constructing relationships between quantities. ***Multiplicative reasoning*** • Reasoning about the multiplicative change in an angle. • Constructing multiplicative relationships between two angles. • Reasoning about angles in relation to a circle. • Constructing composite units of angles.

This study aimed to explore the potential of teaching dynamic angles through the quantitative reasoning lens. Specifically, the goal was to use the instructional tasks for students to bridge multiple angle conceptions. These angle conceptions primarily reflect the geometric, multiplicative, and non-static natures of angles. Consequently, this study seeks to answer the research question: How does teaching dynamic angles through quantitative reasoning help students develop a robust understanding of angles that includes multiple angle conceptions?

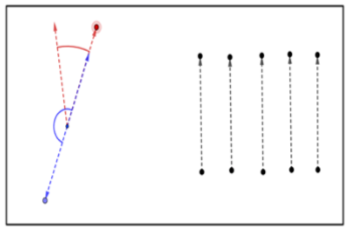
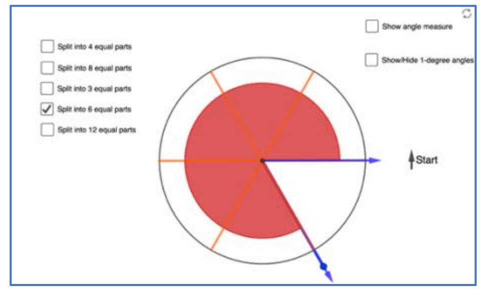
Methodology

Four individual design experiments (Cobb et al., 2003) were conducted with third-grade students to explore how teaching angles dynamically through quantitative reasoning would elicit students' robust understanding of angles by expressing multiple angle conceptions. To do so, a set of digital tasks was designed to reflect the three processes of quantitative reasoning. At each stage of orchestrating quantitative reasoning, the tasks, tools, and questions were used to guide students in exploring dynamic angles, conceiving multiple angle conceptions in generating angles, identifying quantities in generating angles, and constructing relationships between the quantities (see Figure 2). The tasks offered to students are similar throughout each experiment with minor modifications during and before the next experiments. The questions illustrated are semi-structured in that they were flexibly modified to follow up on students' thinking. The modifications of the tasks, tools, and questioning are not discussed in this report.

The participants were Jordan, Angelie, Alicia, and Axel (pseudonyms). Their guardians volunteered them to participate in the study and identified them as on-level in mathematics. All participants were enrolled in different elementary schools in the Northeast United States. They were the only third-grade students who participated in this study. This study focuses on third-grade students because, at this grade level, students begin to learn angles as an attribute of shapes. This also seems to be the appropriate level to explore angles that may prepare students to create holistic angle conceptions useful for understanding other mathematical concepts in the succeeding grade levels.

Figure 2

Sample of Instructional Design for Eliciting Quantitative Reasoning about Dynamic Angles

Stages of Quantitative Reasoning		
Conceiving Multiple Angle Conceptions	Identifying Quantities in Generating Angles	Constructing Relationships between Quantities
 Task: Task that illustrates the union of segments, and when one of the segments are rotated, a wedge is illustrated. Tool: The rotation and tracing tools or shaded sector of a circle. Question: What did you create?	 Task: Compare the two angles Tool: Segments to split the bigger angle and rotation tool Question: What is changing? How do you make an angle bigger/smaller?	 Task: Split a circle into eighths, thirds, and sixths. Tool: Rotation tool, partition lines to show equal split of a full rotation, and wedge to show the amount of rotation. Question: How many degrees does your angle have? How is that size related to the other angle size?

The design experiments were conducted virtually over 4 to 5 sessions and lasted about 45 to 55 minutes each. During each experiment, students were asked to share their computer screens as they interacted with the researcher and the tasks. While the students worked on each task, they were asked to answer the questions to orchestrate their quantitative reasoning as summarized in Figure 1. Each session was recorded, transcribed, and coded. There were two levels of data analyses conducted. First were the ongoing analyses during each experiment, examining students' chronological accounts of reasoning in each task. Then, in the retrospective analysis which took place at the end of all the experiments, students' responses in each task were cross-examined to understand how the instructional design elicits student reasoning. This paper reports the retrospective analysis of the design experiments.

Findings

This paper focuses on the orchestration of the instructional design of tasks, tools, and questions to elicit students' use of quantitative reasoning about dynamic angles. The researcher also highlights the characteristics of the design of tasks to explicate the influence of the characteristics of tasks on eliciting students' reasoning. It is worth mentioning here that students exhibited similar forms of reasoning at different tasks, particularly at the conceiving the angle situation and identifying quantities stages of the quantitative reasoning.

Conceiving the Angle Situation

At the initial stage of quantitative reasoning—conceiving the angle situation—the researcher considered the bridging of multiple angle conceptions in the design of instructional tasks. Specifically, the tasks were designed to illustrate angles as a union of rays through pairs of connected rays or segments. Then, students were asked to move one of the segments or rays to generate angles. The segments or rays as angle sides were designed to only rotate clockwise or counterclockwise when moved by the user. To help students visualize the rotation of these objects through a wedge, the tracing tools leave traces or show a shaded sector of a circle.

While students were rotating the angle sides, they were asked “What did you notice?” The students described that they can “rotate” or “move farther away” the segments or rays to create “a space” between them. Their reasoning showed that they bridged multiple angle conceptions: angles as composed of connected rays, rotations, and wedges. The researcher inferred that the task design that illustrates the geometric components of rays that can be rotated clockwise or counterclockwise, and these rotations are traceable or show the space created by a rotation prompted students to describe angles in terms of multiple angle concepts without using the word angle.

Whenever students mentioned the word “angle”, they were asked to define angles. Jordan and Alicia, although on separate experiments, both described angles as “corners of a shape.” This kind of angle definition is common to students at this grade level probably because this is the most familiar definition they learned in kindergarten and before engaging in angle measure in fourth grade. On the other hand, Angelie’s understanding of angles involves orientations of lines when she made gestures of inclining arms but could not explain it in words. This understanding of angles as the inclination of lines is often difficult even for sixth grade students (Browning et al., 2007). Meanwhile, Axel explained that “an angle is composed of two rays with a common endpoint and those rays can point to different directions.” Both Angelie’s gesture and Axel’s reasoning showed a more sophisticated understanding of angles that are often discussed at higher grade levels. Nevertheless, students’ prior knowledge of angles shows the fragmented conceptions of angles as discussed in the literature. This is in stark contrast to how they reasoned when they were prompted using the designed tasks.

Identifying Quantities when Generating Angles

To prompt students to identify the quantities involved in generating angles when they were rotating the angle sides, they were asked "What are changing?" Jordan rotated a ray to create a quarter of a full turn (represented by a quarter of a circle) and used "90-degrees" to describe the size of the wedge. When I asked Jordan what "degrees" meant, he could not explain it. Jordan showed his preliminary association of a 90-degree angle with the size of a wedge by creating a quarter of a circle, similar to how Confrey et al. (2012) described students connect their conception of a turn with the fraction of a circle, and then with the degree measure. The question potentially prompted him to identify the size of the wedge as the changing quantity. The researcher also inferred that his prior knowledge of fractions emerged due to the design of generating a sector of a circle or a wedge as showing a fraction of a circle.

When Axel was asked how he could make an angle bigger, he explained while rotating one angle side, by "stretching it, like pulling it out." The researcher inferred from his reasoning that he was referring to changing the angle size by pulling one of the angle sides away from the other. Alicia also rotated one segment closer to the other to show that the angle "is smaller." Similarly, Angelie reasoned that "once you move the line farther away from the other line, the angle becomes larger; putting it closer to the other line will make it much smaller…it does not matter the length." Axel, Alicia, and Angelie's reasoning showed that they identified the amount of rotation and the space between the angle sides as the quantities that change when changing the angle size. The design of a non-static feature of the rotation of angle sides and changing the proximity between the angle sides during the rotation may have influenced students to associate these changing quantities as a significant stage of quantitative reasoning.

However, among the four students, only Angelie explained further that the length of the lines does not relate to the size of the angle, either making the angle bigger or smaller. This shows that the design of rotating sides did not elicit the mathematically incorrect association of angle size and the length of its sides found in other studies when students only have a single angle conception (e.g., Keiser, 2004). Students' prior knowledge about angles did not seem to inhibit them from constructing quantities when generating angles via rotation.

Constructing Relationships between Quantities

At the third stage of quantitative reasoning, students exhibit their construction of relationships between quantities when they engage in tasks that show the splitting of a full rotation into a number of equal parts. Students were introduced to understanding a full rotation with 360° during the experiment by exploring a circle split into 360 equal parts. In the subsequent task, students were engaged in splitting a full rotation into eighths. To prompt them in reasoning about how the quantities relate to each other, Axel was asked about the size of an eighth of a turn. He explained, "45 degrees because it is half of 90 degrees." Subsequently, he reasoned that "4/8 is 180 degrees… because it is 2/4, which is 180 degrees." Axel's reasoning about the sizes of the eights of a full turn illustrated a flexible understanding of the amount of turn, associating the fraction of a full rotation with the number of degrees. Similarly, Alicia talked about the number of degrees in an eighth of a full turn, "it is 45…because I separated the 90 into two equal parts. It is 45." The researcher inferred that Alicia utilized the quarter wedge, that she previously created, then decomposed "90 into two equal parts" to claim an eighth as 45 degrees.

When Angelie worked on the same task, she estimated that an eighth "would be 30 degrees… because I learned that one quarter is 90 degrees, which is 2/8. I can't really split it into two equal numbers." Angelie only estimated the degree measure for an eighth of an angle because she could not split 90 into two equal parts. Instead, she created a connection between 1/8 and 90° by stating that 90 was 2/8, or 1/4 in a simplified form. Similar to Angelie's struggle, Jordan thought of an eighth of a turn as "27 degrees." Only when the researcher allowed Angelie and Jordan to use the angle measure tool they were able to find the degrees for an eighth of a full turn and compose 45° and 45° into a 90-degree angle. Compared to Axel and Alicia's use of the wedge that represents the size of an angle and their mental action of splitting a quantity such as half of 90 degrees, this was difficult for Jordan and Angelie who needed to turn the angle measure tool to help them identify the correct value.

Conclusive Remarks

The teaching and learning of a holistic conception of angles have been fragmented as shown in decades of research (e.g., Browning et al., 2007; Clements & Battista, 1989; Confrey et al., 2012) about students' conception of angles and the common learning standards for mathematics. This study shows that designing instructional tasks, tools, and questions that seamlessly bridge multiple angle conceptions may help students develop a more robust understanding of angles. Leveraging the characteristics of the design of tasks, tools, and questions that illustrate the geometric, multiplicative, and non-static natures of angles into a unified construct has been found helpful for orchestrating students' construction of quantitative reasoning about angles. Teachers may modify the design of tasks, tools, and questions necessary to successfully orchestrate students' quantitative reasoning about dynamic angles.

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Framing Reasoning Skills in Secondary Mathematics Learners

VISUAL REPRESENTATIONS PRODUCED BY EDUCATORS IN RESPONSE TO RATIONAL NUMBER TASKS

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An investigation of a sample of Secondary Mathematics Educators responses when asked to create sample visual representations and accompanying explanations for their students based on a set of rational number tasks.

Introduction

One effective method for mathematics instruction is the use of visual representations. The National Council of Teachers of Mathematics (NCTM, 2014) has stated that "Visual representations are of particular importance in the mathematics classroom, helping students to advance their understanding of mathematical concepts and procedure, make sense of problems, and engage in mathematical discourse" (p. 25). Teachers are the conduit for conveying visual representations. They address this task in their lesson planning but must also be ready to answer questions and address misconceptions as they emerge in the classroom. Much of their decision-making draws on their mathematical knowledge for teaching (MKT) which, as outlined by Ball et al. (2008), includes creating and using visual representations to effectively meet student needs. This research study sought to analyze the visual representations that secondary educators produce without prior planning in relation to rational number tasks.

To define visual representations, we draw from Zazkis et al., (1996) and Arcavi (2003). We propose that visual representations are received through the senses, and may consist of pictures, diagrams, or objects presented with an external medium such as paper, a whiteboard, technological tools, or any other external implement used for the purpose of depicting and communicating information to assist a learner's process of internalizing a concept including the mechanics of how it works, how it looks, and any variations (Arcavi, 2003; Zazkis et al., 1996).

Literature

The NCTM identifies five Process Standards deemed necessary for teaching mathematics. Among the standards is Representations (NCTM, 2000). In their seminal work on mathematics education, English and Halford (1995) shared their belief that "The essence of understanding a mathematical concept is to have a mental representation or mental model that faithfully reflects the structure of that concept" (p. 18). Students obtain flexible mental models and the ability to solve more conceptually challenging problems when they first engage with, discuss, and make connections between multiple representations of mathematical concepts (Dreher & Kuntze, 2014; NCTM, 2014). Representations can take many forms and may possess attributes that dictate which concepts they are best suited to represent.

Visual representations are often associated with a lower form of instruction aimed at those in their early education, while secondary and post-secondary students are expected to use more sophisticated symbolic and linguistic forms of representation (Alsina & Nelson, 2006; Arcavi, 2003). Instead, Arcavi (2003) suggests visuals play a key role in reasoning, problem solving, proving, and obtaining results. Within secondary education, classroom teachers are the most important source for providing students with opportunities to view, experience, and create visual representations. Fortunately, there have been technological developments to assist teachers in providing visual representations. Graphing calculators and online dynamic graphing platforms such as GeoGebra or Desmos provide students with tremendous experience manipulating graphs and equations (Liburd & Jen, 2021; Sinclair et al., 2012; Thomas & Hong, 2013). However, not all visual representations can or will be presented through technology. As needs emerge, teachers spontaneously provide clear and complete examples. Researchers have noted the need for visual representations in secondary mathematics that contain complex figures (Cox & Lo, 2012). For example, when speaking of geometric shapes, Cox and Lo (2012) reported the need for students to have experience with not just simple figures, but complex figures. They defined *complex figures* as a group that "excludes all shapes commonly found in a textbook" (Cox & Lo, 2012, p. 32). This points to the need for secondary teachers who can create and use visual representations effectively, which has implications for pre-service programs and for in-service professional development.

Classroom teachers plan if and which representations will be used in class, as well as to what degree visual representations will be incorporated into the mathematical experiences students have (Ball et al., 2008). In addition to planning, research suggests that a teacher's ability to notice and address a student's conceptual needs will depend on their knowledge of both content and students (Ball et al., 2008; Dreher & Kuntze, 2015; English & Halford, 1995). These typical teaching tasks go far beyond general math ability (Ball et al., 2008). Speaking of MKT, Ball et al. (2008) suggested that teachers "must hold unpacked mathematical knowledge because teaching involves making features of particular content visible to and learnable by students" (p.

400). Ball et al. (2008) acknowledged specialized content knowledge (SCK), a domain within MKT, needed for teaching which includes anticipating common student misconceptions, recognizing different interpretations of concepts, and making effective representations to help students with conceptual understanding. It is upon this last task which we focus this study.

There is a lack of current research investigating the ability of mathematics teachers to produce visual representations without preparation but as classroom instruction may necessitate. To contribute to the research, we will ask in-service teachers to spontaneously create visual representations for tasks involving rational numbers. The research question we pursue is: What visual representations and explanations will in-service secondary educators create and use when asked how they would demonstrate various rational number operation tasks for their students?

Methods

The participants in this study consisted of three female secondary mathematics educators in a large school district in the South-Central United States. All three educators were traditionally certified in their state as opposed to alternatively or emergency certified. A traditional certificate is a result of completing a state-approved education program which includes an internship. Dee was serving as an on-level 6th Grade Mathematics teacher. She had 13 years of experience in both middle and high school spanning a variety of mathematics courses. Pat was serving as an on-level Geometry teacher. She had 13 years of experience in middle and high school covering a variety of mathematics courses. Lisa was serving as a 12th Grade Pre-Calculus teacher/facilitator with ties to the local community college program. She had a degree in mathematics with a minor in teaching which included a teaching internship. She had 25 years of experience in both middle and high school teaching a variety of mathematics and computer courses.

Research Design and Data Collection

No mathematical ability pre-test was administered, however each educator expressed confidence in their general mathematical abilities. The educators were interviewed in their classrooms. This was intentional as it would allow each participant to be in the setting and frame of mind where they generally address tasks like the ones presented in the interview. Each educator was asked to create a visual representation for 11 rational number tasks and share how she would explain it to students. The same 11 tasks were presented to each educator (examples of tasks not reported below: $\frac{3}{4}, \frac{12}{7}, \frac{3}{4}+\frac{1}{2}, \frac{2}{3}-\frac{3}{5}$). Interviews were video recorded and transcribed. All drawings and/or symbolic work was collected.

Data Analysis

Educators' responses for each task were coded and evaluated inductively based on characteristics of the representations (what kinds of models and how parts were indicated), ability to create representations, and explanations of operations. The author and a colleague collaborated to consider SCK by scoring the visual representations and verbal explanations. For each task, we considered clarity and completion when coding. Each visual representation (VR) was rated as either unable to start (1), partial completion (2), completed but weak/lacking in clarity (3), or completed strong and clear (4). Each verbal explanation (VE) that accompanied the representation was rated as either not connected (1), relevant but weak connections (2), relevant with connections made (3), or relevant and easily adapted to address misconceptions (4). Adding these scores became an overall score. These ranks were minimal (1-2), weak (3-4), moderate (5-6), and strong (7-8). These ranks are not part of a validated instrument, but a way to analyze the findings and consider relationships between the representations created and the educators' MKT.

Findings

When asked about the frequency of use of visual representations in their instruction, each educator indicated a high frequency of use. Each expressed a belief that visual representations are valuable and sought to incorporate a variety for their students. All three educators felt that rational number operations were common occurrences in their courses, however, Dee felt a stronger tie since the skill is part of the $6^{th}/7^{th}$ grade curriculum in her district.

We found that Lisa was able to produce visual representations for static fractions, but when operations were introduced, she was only able to produce algorithmic representations for each task. Lisa felt confident that she could produce visual representations if provided with time to research the best methods, but it was not a skill she used in her Pre-Calculus instruction, and she could not produce any examples spontaneously. Her algorithmic representations were all completed accurately. Since her answers are not examples of visual representations, we will not review her work in this article, but suggest that her MKT for rational number operations had not been exercised as part of her current teaching duties and, as a result, was not readily accessible.

In our characteristics analysis of Dee and Pat's work, we found their work included bars, number lines, circles, area models, blocks, eggs, pennies, and boxes. Shading was the most common method to indicate the part of a whole needed for the task, followed by drawing partition lines, drawing arrows, and circling relevant pieces. They both mentioned using color as they worked, indicating the layers of information they wished to convey.

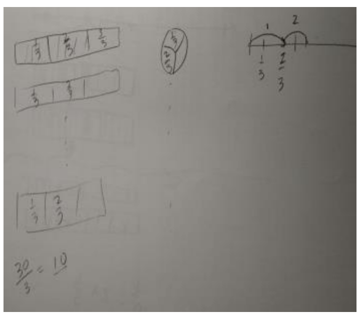
The visual representations and explanations outlined in the next section are the educators' responses to Tasks 6, 7, 9 and 10. Each task is provided below exactly as it was presented to the educators. Each educator's task report is followed by the assigned clarity and completion code.

Task Comparisons

Task 6: How would you create a visual representation of $15 \times \frac{2}{3}$ and explain the operation?

Figure 1

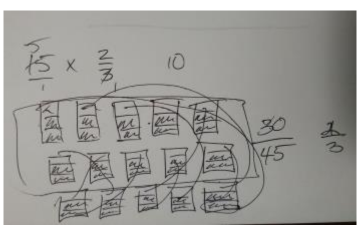
Dee's Task 6 Representations



Dee created three visual representations (Figure 1). First, she drew a bar divided into thirds. She shaded 2 of the thirds and indicated redrawing this bar 15 times (she used dots to indicate iterations). She explained that $2 \times 15 = 30$. "So you've got 30 thirds or 10." She moved into a circle version and a number line version. Each time iterating a model of $\frac{2}{3}$. (VR-4; VE-4; Overall-strong)

Figure 2

Pat's Task 6 Representations

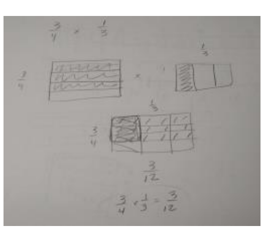


Pat drew 15 rectangles and shaded $\frac{2}{3}$ of each (Figure 2). She then counted 30 shaded pieces out of 45. There was a look of confusion on her face when she realized that $\frac{30}{45} = \frac{2}{3}$. She paused and then changed her strategy to assembling the shaded pieces into new units of three which amounted to 30 thirds or 10 whole rectangles. Satisfied, she commented that she might use stacks of pennies with her students. (VR-3; VE-3; Overall-moderate)

Task 7: How would you create a visual representation of $\frac{3}{4} \times \frac{1}{3}$ and explain it?

Figure 3

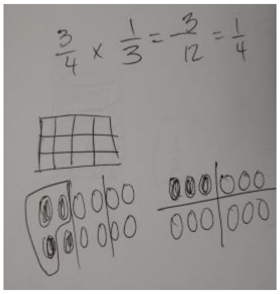
Dee's Task 7 Representations



Dee quickly drew an area model for $\frac{3}{4}$ side-by-side with another one for $\frac{1}{3}$ (Figure 3). She redrew them into one overlapping model saying, "Multiplication is overlap". She explained how this new area model displayed $\frac{3}{4}$ of $\frac{1}{3}$. Dee commented it is “harder to show simplification” in models. She noted that 6th graders don’t often make the connection. (VR-4; VE-4; Overall-strong)

Figure 4

Pat’s Task 7 Representations

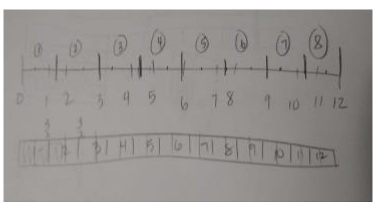


Pat stopped and contemplated “Something you can make into quarters and thirds. Let’s go with a dozen eggs.” She drew a dozen eggs, divided them into thirds and identified $\frac{3}{4}$ of one third (Figure 4). She continued by demonstrating that if she divided the eggs into fourths instead and found $\frac{1}{3}$ of a fourth, the answer would be the same. (VR-4; VE-4; Overall-strong)

Task 9: How would you create a visual representation for $12 \div \frac{3}{2}$ and explain the operation?

Figure 5

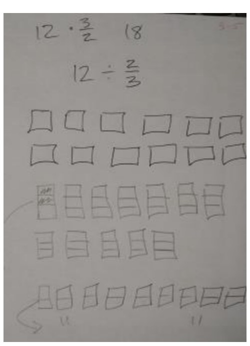
Dee’s Task 9 Representations



Dee used a number line in her first attempt (Figure 5). She said, “I’m going to try to get them evenly spaced” as she ticked off $\frac{3}{2}$ markers. This method worked. She then showed how a bar model would work similarly. When asked if she could use an area model, Dee said, “I don’t know. I haven’t done division with those. I think you’d have to pull it apart because it starts with 12.” (VR-4; VE-4; Overall-strong)

Figure 6

Pat’s Task 9 Representations

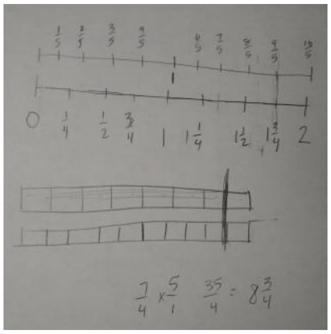


Pat drew 12 squares and said, “I’m trying to think how I can make more, it should be more.” She decided each item should be $\frac{2}{3}$ of the original box but wasn’t sure how that would make more. She said, “My new ones aren’t as big as the originals, but I have more of them.” She struggled to find a logical path. Ultimately, her visual representation showed how to divide by $\frac{2}{3}$ rather than $\frac{3}{2}$. She was unsuccessful. (VR-2; VE-2; Overall-weak)

Task 10: How would you create a visual representation for $1\frac{3}{4} \div \frac{1}{5}$ and explain the operation?

Figure 7

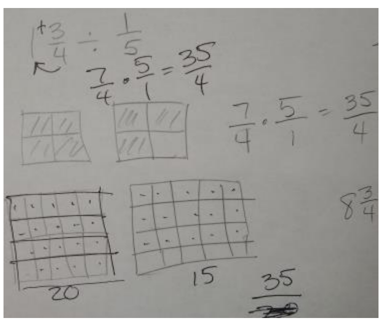
Dee's Task 10 Representations



Dee was able to compare fourths and fifths well enough to reach the whole number part of the answer but was not able to show the remaining fractional answer. She said, "I don't know if this is going to work. I'm kind of making it up as I go . . . Division is hard I think. It's hard for me to visualize." She mentioned breaking the number line up into 20ths to see it, but then drew a bar model in fourths and fifths like the first attempt. (VR-2; VE-2; Overall-weak)

Figure 8

Pat's Task 10 Representations



Pat drew a 4 x 5 area model and arrived at the number 35 but was unsure if it was $\frac{35}{4}$ or $\frac{35}{20}$. Pat resorted to working on the problem algorithmically and then went back to the representation. She said, "I don't know. That one's tough. I'm thinking you'd have to go for something that has 20 in it. I'd have to spend more time with this one." (VR-2; VE-2; Overall-weak)

Discussion

We sought to analyze the visual representations and explanations in-service educators would create and use as they attempted each task. We synthesize our observations thus:

Visual Representations: Dee consistently modeled multiple visual representations for each task which were strong and clear. Her only stumble was Task 10. Pat gave extra effort to finding realistic situations on which to base her visual representations (ex: dozen eggs). However, her visual representations were busy and lacked the clarity that would help students' conceptual understandings develop.

Explanations: Dee was extremely adept at explaining an operation's meaning and its effects on rational numbers, effects which are often quite opposite to the effects on whole numbers. Her verbal explanations were mostly relevant and easily adapted to address misconceptions. Pat often spoke of connecting common denominators to the tasks but did not provide accurate accounts of how to garner the final answer from her models. Interestingly, her division task strategies were appropriate, but she was unable to talk her way through to completion. We have two suggestions for ways to strengthen educators' ability to create and use visual representations for rational number instruction. First, more specific pre-service training and/or professional development for in-service educators based on more nuanced research could help fill the gaps discovered in this and other related research. As Ball et al. (2008) suggested, preparation must go beyond general math ability to address making content visible and learnable by students including anticipation of misconceptions and recognizing different interpretations. Second, the creation of an easily accessible resource filled with visual representations for all types of rational number operations with explanations on how to use each one. Future research could include a more systematic way to capture educator's knowledge of rational number operations in relation to visual representations of those operations. These suggestions are meant to assist mathematics educators who, like our participants, agree that visual representations are valuable in mathematics classrooms.

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IMPROVEMENT IN MATH PROBLEM SOLVING IS MODERATED BY WORKING MEMORY

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The purpose of this study was to investigate whether middle school students' working memory capacity influenced the impact of an intervention aimed at improving their problem solving proficiency. The sample included a total of 179 grade 6 and 7 students from a middle school located on the West Coast. The results suggest that gains from the problem solving intervention were moderated by working memory capacity, with students with higher initial working memory capacity showing the largest gains on problem solving proficiency.

Background

Gaining proficiency in problem solving (PS) is a key outcome in K-12 mathematics (NCTM, 2014). Proficiency in PS—operationalized herein as the ability to successfully solve cognitively demanding mathematics word problems—has been shown to be related to numerous factors such as metacognition, executive function (EF), content knowledge, strategic thinking, and affective characteristics such as student beliefs (Chapman, 2015; Rhodes et al., 2023; Schoenfeld, 2013). Specifically, PS requires that students decode tasks, transpose problem information using mental models, process information, and implement plans (Singer & Voica, 2013), all of which involve cognitive (particularly EF) processes. However, despite increasing evidence of the various cognitive and affective factors that influence PS, little remains known about how best to improve PS performance in students (Lester & Cai, 2016).

Working memory (WM), considered a keystone EF ability (Friedman & Miyake, 2017), has been implicated as a critical factor in mathematics (e.g., Bull & Lee, 2014; Raghubar et al., 2010). WM is a mental workspace used to maintain short-term focus of attention and manipulate this information, often in the service of accomplishing complex cognitive processing (Baddeley & Hitch, 1974). A relationship between WM ability and academic achievement comes from both theoretical accounts (e.g., Miyake & Shah, 1999), as well as empirical studies demonstrating correlations between mathematic performance and WM ability, including meta-analyses (e.g., Friso-van den Bos et al., 2013) and longitudinal studies (Alloway & Alloway, 2010). EF skills can depend on contextual factors, and with appropriate training and scaffolding can be developed and strengthened.

Naturally then, it has been posited that strengthening WM might in turn improve math performance. However, much is still unknown about the precise contribution of WM to math, and in particular to mathematical PS proficiency. This includes whether, and how, variations in WM may affect targeted math interventions. Understanding these connections can inform future math pedagogy, such as whether differentiated WM support is needed. Thus, the purpose of the present study was to investigate the role of WM capacity as moderator of gains in mathematical PS, in the context of a larger PS intervention study. Guiding this work, we posed the following research question: Does an individual's baseline working memory capacity affect their gains in mathematical problem solving performance?

Theoretical Framework

Working memory is a critical factor in many theories of information processing and cognition (e.g., Anderson et al., 1997; Meyer & Kieras, 1997). It has been linked to the ability to focus attention, in the face of distractors, to important information and details relevant to one's current goal, especially in the formation of new concepts and how multiple concepts relate–such as is required for mathematics (or any) learning (Cowan, 2014). This idea is further supported by research demonstrating a relationship between individual variation in working memory capacity (the number of items one can retain) and mathematics ability (e.g., Friso-van den Bos et al., 2013; Raghubar et al., 2010). Additionally, evidence suggests that developing EF skills, including WM, support the development of math learning and problem-solving, and vice versa (Clements et al., 2016; Zelazo et al., 2017). Recent evidence also supports the *combined* importance of metacognitive ability and EF (along with student beliefs and prior content knowledge) in the support of proficient PS (Rhodes et al., 2023). Yet little is known how individual differences in WM ability may impact math PS proficiency.

Description of Intervention

The present study was part of a larger study which aimed to improve mathematical PS performance in middle school students. In this larger study, students in an intervention group used a PS application that scaffolded and targeted EFs and metacognition within a four-phase attack strategy that was based on the work of Pólya (1945/2014). Scaffolds and supports included breaking the problem down into the four phases noted above, asking students what they notice and wonder about the problem, prompting students to explicitly consider what the problem was asking them to do, having students journal their plans for solving the problem while providing them with sample sentence stems, and having students explain and record their solution. The results come from a larger Pre vs. Post assessment (separated by about five months) study which showed that the intervention group significantly improved on mathematical PS performance when compared to students in a business-as-usual control group, as measured by the PS measure described below (see Rhodes et al., in preparation). Thus, the purpose of the present study was to expand this work by exploring whether the gains seen in the intervention group were moderated by working memory.

Methodology

Participants

The participants in the study were 6th and 7th grade students from a single school, referred to herein as Beach View. Beach View Middle School is located in a large, suburban school district from a West Coast State. All mathematics teachers at the school were offered the chance to participate in the study, along with all students enrolled in classes taught by participating teachers. Of these students, 92 6th grade students and 87 7th grade students completed both measures and are included in the analyses reported herein. The students self-identified as girl (n = 103), boy (n = 66), non-binary or prefer to self-identify (n = 5), prefer not to say (n = 3). Students self-identified (note multiple categories could be selected) as African-American or Black (n = 19), Hispanic, LatinX, or, Mexican (n = 131), Asian (n = 19), Chaldean or Middle Eastern (n = 69), Native American or Alaska Native (n = 5), Pacific Islander (n = 3), White (Non-Hispanic; n = 17), self-identified (n = 34), or prefer not to say (n = 33).

Measures and Scoring

Executive Function. The Adaptive Cognitive Evaluation (ACE;) was used to measure students' EFs; evidence supporting ACE as a valid measure of EF is presented in existing literature (Younger et al., 2022). The ACE is comprised of gamified, computer-based versions of well-known tasks that measure core EFs such as working memory, cognitive flexibility, and inhibitory control. Within the present study, working memory was the variable of interest and was measured using a change detection task (Luck & Vogel, 1997), with the key measure of interest being "K," an estimate of one's visual-spatial working memory capacity (i.e., the number of visuo-spatial items one can hold in mind at one time). K was calculated using the standard formula computing hits minus false alarms in the set size 2 condition, thus scores can range from

0 to 2. The set size 2 condition was used given average performance in the set size 4 condition fell below chance levels of performance.

Problem Solving Measure. Problem solving was measured using a 3-item test that consisted of problems that were written by Illustrative Mathematics (IM) and that was administered outside of the application used as part of the intervention. The items were chosen based on three criteria. Specifically, the problems 1) had a high degree of cognitive demand as assessed by the Smith and Stein (2018) framework; 2) align to priority standards within the district's pacing guides; and 3) offer opportunities for students to show or explain their process for solving the problems. If a selected problem did not offer sufficient opportunities to illuminate students' thinking, slight modifications were made to the directions of those problems. Given that the intervention was aimed at supporting students in learning the process of problem solving rather than any specific content or type of problem, problems were not explicitly aligned to any aspect of the intervention outside of the three aforementioned criteria. Each students' work was scored two ways: accuracy (total correct solutions) and understanding (the level of correct relevant mathematical thinking that the student demonstrated, regardless of answer accuracy). Fleiss' kappas were calculated to measure interrater agreement on the understanding scoring: .961 and .703 for 6th grade and .880 and .842 for 7th grade, for accuracy and understanding, respectively. This alignment to the problem-solving framework and interrater agreement provides evidence of validity and reliability related to the PS measure used.

Data Analysis. To start, 1-tailed Pearson correlations were used to examine the relationships between WM baseline (Pre-Test) scores and student scores on the PS measure. Given individual differences in WM have been shown to correlate with academic performance, including general mathematic ability, we were interested in examining whether changes in problem solving were moderated by students' baseline working memory score, we first computed median K (WM capacity) scores for 6th grade and 7th grade independently (to account for presumed developmental differences across the grades). A median split procedure was used, for each grade, such that students were split into "below-median" and "above-median" WM groups, using their PRE K score. Separate two by two multivariate analyses of variances (MANOVA) were calculated for the IM Accuracy and IM Understanding dependent variables, using the factors of wave (Pre vs. Post assessment) and the between-subjects WM group factor (low vs. high).

Results

Pearson correlations were low and non-significant when comparing WM baseline (Pre-Test) to pre-test PS scores (n = 179) with r = -.019, p = .399 for accuracy, and r = .056, p = .229 for understanding. However, the correlations comparing WM baseline (Pre-Test) scores to PS scores on the post-test were stronger, and significant, with r = .157, p = .018 for accuracy, and r = .207, p = .003 for understanding. In addition, we found small and significant correlations between PRE K scores and improvement in IM Accuracy and Understanding (Post minus Pre scores, on each measure, r = .150, p = .045 and r = .188, p = .012, respectively. At the multivariate level, Mahalanobis Distance was used, and one outlier was noted and removed and the MANOVA was re-run. The final results are reported below.

Figures 1 and 2 show mean scores by wave (pre vs. post) and WM group for IM Accuracy and Understanding, respectively. Higher scores for POST compared to PRE ("intervention effect") were observed, with this difference being more pronounced in the group with higher PRE WM capacity (K) scores, in both the IM Accuracy and Understanding measures. The interactions for wave by WM group were significant, for both IM Accuracy $F\ (1,176) = 6.113$, $p = .014$, $\eta_p^2 = .034$ and IM Understanding $F\ (1,176) = 8.042$, $p = .005$, $\eta_p^2 = .044$.

Table 1.

Mean and Standard Deviation Scores for IM Accuracy and Understanding

	N	Pre: IM Accuracy	Post: IM Accuracy	Pre: IM Understanding	Post: IM Understanding
Below Median K Group	92	.674 (.64)	.842 (.61)	.348 (.34)	.553 (.41)
Above Median K Group	86	.657 (.67)	1.11 (.62)	.414 (.33)	.811 (.54)

Figure 1.

IM Accuracy Scores by WM group and Time (Pre vs. Post)

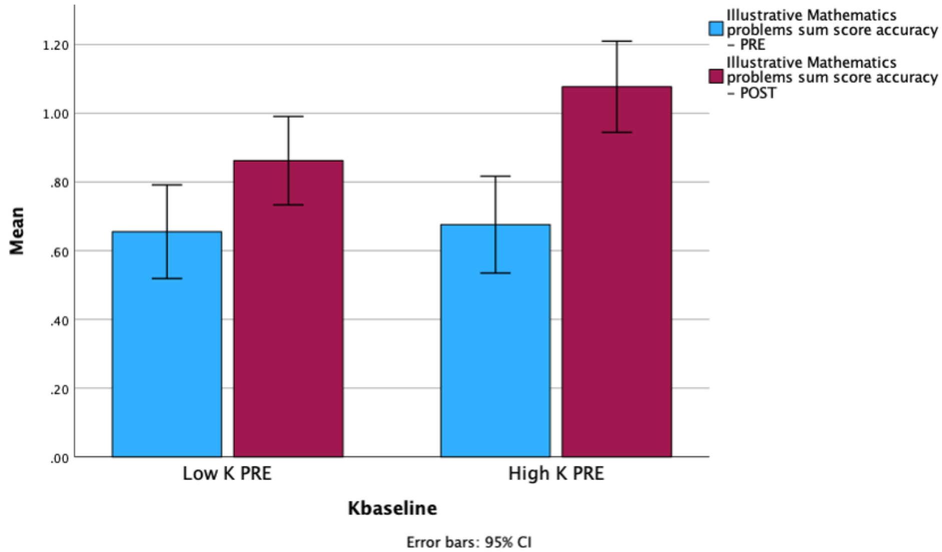
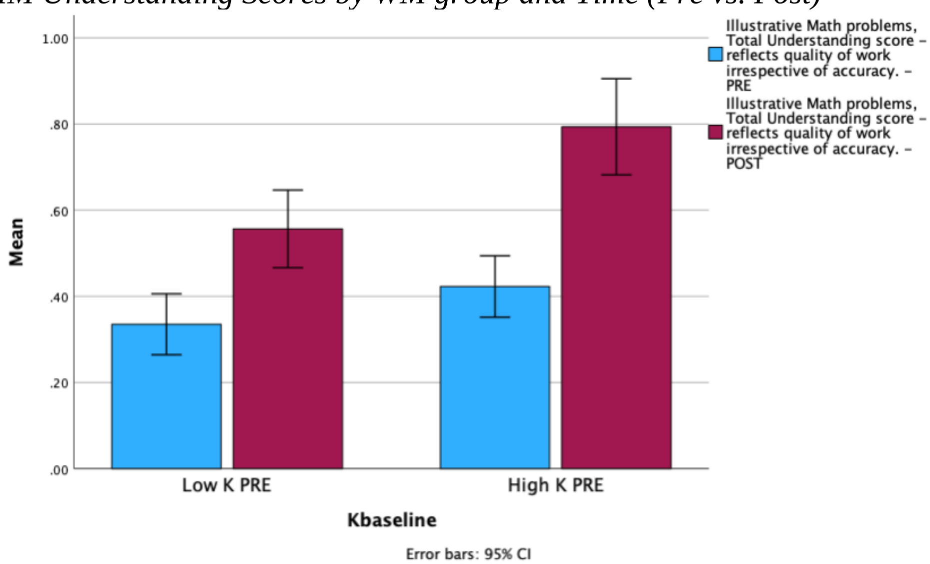


Figure 2.

IM Understanding Scores by WM group and Time (Pre vs. Post)



Discussion

The purpose of the present study was to explore whether students' initial WM capacity moderated the impact that a PS intervention (aimed at all students) had on students' PS performance. The results suggest that students' initial working memory capacity moderated the effectiveness of the PS intervention. Although students in both the below-median WM group and students in the above-median WM group significantly improved their PS performance across accuracy and understanding, that improvement was significantly higher for students in the above-median WM group. Significant correlations between WM scores and gains in PS strengthens this argument.

In seeking to interpret these results, it is important to note that baseline WM was not correlated to PS scores at pre-test, but was correlated to PS scores at post-test, with students in the above median WM baseline group showing significantly more growth. In addition, prior research suggests that problem solving involves numerous cognitive processes such as decoding data, creating mental models, and applying techniques to solve problems (Singer & Voica, 2013) – all of which are likely to put a high demand on working memory. Taken together, we theorize that students with higher WM capacity may have more effectively encoded, and then retrieved, the intervention scaffolds when needed from long-term memory. When the WM demands of doing so are taken into account, in conjunction with the cognitive demands inherent to problem solving, it stands to reason that students with a high WM capacity would be better equipped to retrieve and utilize the scaffolds when they were no longer being explicitly provided to them. Although students in the below-median WM group still improved their PS performance, the fact that they improved less than students in the above-median WM group may suggest that they were able to retain and utilize only a subset of the scaffolds, and/or were less effective in applying those scaffolds outside of the intervention itself.

Students with higher WM may also be better equipped to deal with the relatively high cognitive load (WM demands) inherent in the PS intervention platform. In other words, they were better able to process the multitude of information presented in each of the four-phases of the intervention program and thus better utilized the embedded scaffolds. This could also explain the results presented here, either alone, or in conjunction with the above explanation about transfer of the scaffolds to situations where they were not present.

These results have broad implications for classroom instruction related to mathematical PS in the middle grades. Specifically, they provide evidence that WM is critical to consider when designing PS interventions. The results may also suggest that teachers and researchers need to explicitly consider how to support students in retaining and transferring WM scaffolds beyond intervention conditions to ensure that they can apply these skills to other class activities and assessments. Teachers should be cognizant of varying levels of WM capacity in their students and provide EF scaffolds and supports in the moment. This will in fact inform future iterations of the broader intervention, such that additional supports will be provided to reduce cognitive load and WM demands during different phases of the program.

General approaches include awareness of extraneous load in math problems (e.g., overly complex, or unneeded wording), build in more time to process problems, and add prompts for taking action and/or reflection (e.g., "what is the next step I should take in solving this problem?"). In short, teachers should consider ways in which extraneous WM demands can be lessened without changing the rigor of problems. In conclusion, the results suggest that WM is an important variable to consider in for improving PS proficiency in middle school students.

Limitations and Avenues for Future Research

There are several limitations regarding the current study. First, students' WM was measured at a single point in time and was, therefore, treated as a trait-based variable. However, new research has suggested that EFs may be state-based. Thus, future studies should consider utilizing in-the-moment measure of EFs or measuring EFs at numerous points during studies rather than just at pre- and post-test. Secondly, the design of the present study limited the researchers' ability to explore causal effects and thus future studies may consider how to gain more nuanced understandings of these relationships through true experimental designs and/or the use of qualitative methods such as cognitive interviews. Other future studies could explore the efficacy of differentiating the types of EF scaffolds given during mathematics PS instruction based on individual student's WM ability.

Acknowledgements

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COMPUTER ADAPTIVE MATHEMATICAL PROBLEM-SOLVING MEASURE: A BRIEF VALIDATION REPORT

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The purpose of this proceeding is to share a component to a validity argument for a new, computer adaptive mathematics Problem-Solving Measure that is designed for grades six through eight (PSM 6-8). The PSM is a single test, which uses computer adaptive features to measure students' performance using instructional standards. It is intended to measure students' problem-solving performance related to instructional standards.

Introduction

In prior studies (e.g., Bostic & Sondergeld, 2015; Bostic et al., 2017; Bostic et al., 2021), our research team described a series of vertically equated, paper-and-pencil measures of mathematical problem solving for grades 3-8. The measures draw upon guidance from the *Standards for Educational and Psychological Testing* (*Standards*; AERA et al., 2014), and align to the Common Core State Standards for Mathematics (CCSSI, 2011) for both content and practice. The tests filled a need for K-12 educators, researchers, and evaluators. K-12 educators recently requested, partially due to their experiences during COVID-19, a version that could be administered online, and could be adaptive to students' abilities. This presented an opportunity to develop a computer-adaptive version of the tests. This study's purpose is to present validity evidence based on test content and consequential (i.e., bias) considerations for a computer-adaptive test (CAT) of mathematical problem-solving for grades six through eight (CAT PSM 6-8).

Relevant Literature

Problems and Problem Solving

Our research team drew upon two related frameworks for mathematical problems. Schoenfeld (2011) frames a mathematical problem as a task presenting to an individual such that (a) it is unclear whether there is a solution, (b) it is unknown how many solutions exist, and (c) the pathway to the solution is unclear. Verschaffel and colleagues (1999) frame mathematical word problems as tasks that are (a) open, (b) complex, and (c) realistic. Open tasks can be solved using multiple developmentally-appropriate strategies. Complex tasks are not readily solvable by a respondent and require productive thinking. Notions of open and complex are clearly related to Schoenfeld's framing of problems. The use of realistic adds a necessary element to effectively frame word problems for our assessment. Realistic word problems draw upon real-life experiences, experiential knowledge, and/or believable events. Schoenfeld (2011) and Verschaffel et al.'s (1999) frameworks provided our team with sufficient grounding to develop mathematical word problems for the CAT PSM 6-8.

Given our selection of two synergistic frameworks for creating CAT items, we retained Lesh and Zawojeski's (2007) problem-solving framework from past test development. That is, problem solving is a process of "several iterative cycles of expressing, testing and revising mathematical interpretations – and of sorting out, integrating, modifying, revising, or refining clusters of mathematical concepts from various topics within and beyond mathematics" (Lesh & Zawojewski, 2007, p. 782). Through these frames of mathematical word problems and problem solving, our team sought to develop the CAT PSM 6-8.

Validity and Validity Arguments

Validation studies are intended to provide a reader with information about how evidence supports an intended interpretation and use of test results (AERA et al., 2014; Carney et al., 2022; Kane, 2006, 2012). The *Standards* defines validity as "the degree to which evidence and theory support the interpretations of test scores for proposed uses of tests" (AERA et al., 2014, p. 11). The *Standards* describe five sources of validity evidence: test content, response process, internal structure, relations to other variables, and consequences of testing. More information about these five sources is discussed in Folger et al. (2023). This proceeding focuses on test content and consequential evidence to explore three validity claims. The first claim is that the CAT PSM 6-8 items address mathematics content described in the CCSSM and have a known mathematical solution space. Our second claim is that CAT PSM 6-8 items adhere to the open, complex, and realistic framework. A third claim is that CAT PSM 6-8 items possess limited bias. Our broad research question is: What test content and consequential evidence supports our claims regarding CAT PSM development?

Method

A design science approach (Lesh & Sriraman, 2005; Middleton et al., 2008) was used to develop the CAT PSM 6-8. Design science research is valuable for creating products like tests that can be evaluated, refined, and re-evaluated (Middleton et al., 2008). Our test item development and subsequent validation process had multiple checks and balances across a diverse research team including mathematics educators, assessment scholars, psychometricians, and graduate students across all three areas. Recent scholarship indicates a deductive, a priori, approach for developing validity claims and then collecting validity evidence leverages the theoretical foundation from which the test is developed (Folger et al., 2023); hence, we use that approach in this study. Note, this study focuses on validity evidence based on test content and consequential considerations (i.e., bias) because (a) of similar data collection and analysis method and (b) these claims can be described within the proceedings page limitations.

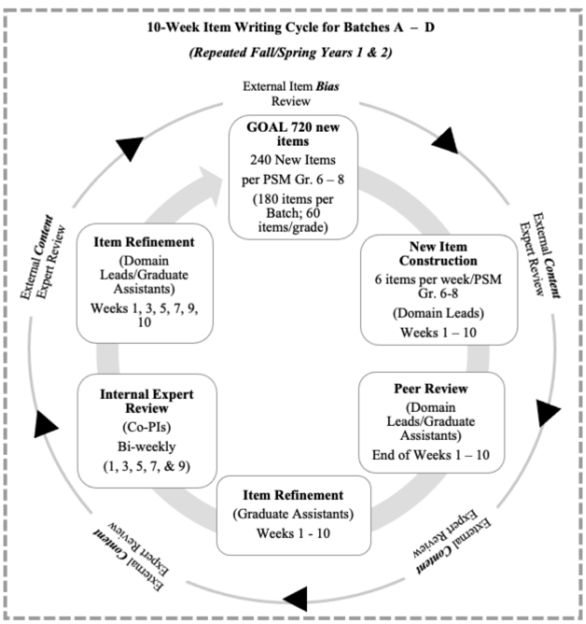
Participants and Instrumentation

CAT PSM 6-8 items were developed using an iterative design (see Figure 1). Item developers included mathematics teachers across the USA certified to teach grades 6-12. All teachers hold graduate degrees and valid teaching credentials in their state. Content panel experts included two terminally degreed (PhD) mathematics educators, mathematics education graduate students, as well as two terminally degreed (PhD) mathematicians who had expertise with instructional standards (e.g., CCSSI, 2011). Bias panel experts included two terminally degreed mathematics educators, an assessment scholar and doctoral student, as well as four purposefully recruited mathematically focused bias panel members representing diversity, equity, and inclusivity through different ethnicities, cultural backgrounds, and geographic regions of the USA. All bias panel members hold graduate degrees and work in education or education-adjacent fields (e.g., engineering). Also, students from participating districts were asked to review and respond to two questions following their work on three CAT items during 1-to-1 think alouds: (1) Do you believe this item is appropriate for other students in your grade level? (2) Do you feel there is any form of bias in the item that you completed? Participating school districts included over 500 diverse students from the Midwest, Southwest, and Mountain West regions, including rural, suburban, and urban school contexts. One group of students from the Mountain West region included a large number of multi-lingual learners.

A goal of CAT PSM 6-8 development was to develop 240 items for each grade level (grades six, seven, and eight). After the multilevel reviews during the item development phase (see Figure 1), the final item pool consisted of a total of 178 items, 178 items, and 182 items all aligned to their respective grade-level mathematics content standards. An example seventh-grade Expressions and Equations item is provided to contextualize the word problems created for the CAT PSM: "A water tower contains 16,880 gallons of water. Each day half of the water in the tank is used and not replaced. This process continues for multiple days. How many gallons of water are in the 4tower at the end of the fourth day?" One example of a reason that an item did not make the final pool of items was that some items could not be further shortened in text length without losing realism and complexity. Similar to past paper-and-pencil PSMs, CAT PSM 6-8 items are scored dichotomously as correct or incorrect.

Figure 1.

CAT PSM 6-8 item writing cycle process



Data Collection and Analysis

This study focuses on test content and consequential validity evidence; further validity claims and evidence will be presented in future scholarship. Gathering evidence based on test content involved multiple stages of data collection, which were analyzed and reviewed in light of each other to triangulate findings (Saldaña, 2013). Data sources are described in the order conducted. One source came from middle school teachers' reviews. Teachers were instructed to review items to confirm that each item (a) addressed the mathematics content and practice standards indicated by the item writer (CCSSI, 2011), (b) adhered to our selected frames for mathematical word problems (i.e., open, complex, and realistic), and (c) were developmentally appropriate. A second data source came from the mathematicians who conducted reviews to explore the degree to which items (a) addressed the mathematics content and practice standards indicated by the item writer (CCSSI, 2011), (b) adhered to our selected frames for mathematical word problems (i.e., open, complex, and realistic), and (c) had a known mathematical solution space. A third data source was a review conducted by two mathematics educators and multiple mathematics education graduate students. That review paralleled prior reviews in that it combined elements from the mathematicians' and the mathematics teachers' reviews. A fourth data source came from students' feedback during think alouds regarding grade-level appropriateness of items.

Bias was reviewed in a similar fashion but with different individuals. Again, the data sources order coincides with the steps in the process. The first data source was a review conducted by practicing mathematics teachers. The goal was to explore the ways in which items might contribute negative bias towards students' outcomes. A second data source came from a review conducted by a bias panel led by a psychometrician and assessment-focused graduate student. This panel followed a protocol developed by the team, based upon past work on a similar project. A third data source came from a review conducted by two mathematics educators, two assessment scholars, and several mathematics education graduate students. This team reviewed items for potential bias, feedback was shared, and revisions were made. A fourth and final data source was students' responses to the bias question asked during 1-1 think alouds.

Data sources were analyzed qualitatively for evidence in support of conjectured claims. Our team used an iterative, inductive analysis with a goal of generating themes (Hatch, 2002; Saldaña, 2013). Step one was becoming familiar with the available data for analysis. Step two was to more closely examine data sources to clarify any ambiguity that arose during the first review of data. Step three was making notes about potential ideas that seemed relevant to the claims, based upon the data sources for each validity source. Step four aimed to categorize notes into general notions, which had potential to become a claim. Step five was discussions about categories that might be eliminated or revised based upon the findings occurred. Step six was to review each category and consider the amount and quality of evidence related to it, which made generate a validity claim. Those categories with two or more pieces of counterevidence or a paucity of evidence were removed. Step seven involved synthesizing those categories into support for the a priori validity claims.

Findings

Test content and consequential validity evidence are presented in relation to validity claims. Based on the study findings three validity claims were generated: (1) CAT PSM 6-8 items address mathematics content described in the CCSSM and have a known mathematical solution space; (2) CAT PSM 6-8 items adhere to the open, complex, and realistic framework; and (3) CAT PSM 6-8 items have evidence of limited bias.

Claim #1: Mathematics content

There was consistent and resounding support that the final drafts of items were aligned to CCSSM standards and had a known mathematical solution space. Closure was important because if there were multiple solution sets, then scoring dichotomously could be problematic. Regarding standards alignment, initial reviews during the item writing stages flagged certain items for further revisions. Flagged items were revised and demonstrated full content alignment with standards. As one example, one review on an Expressions and Equations item suggested that the item changed from "the most" to "the greatest number". The latter statement is mathematically accurate whereas 'most' does not necessarily imply larger numbers.

Claim #2: Open, complex, and realistic

Content panel members consistently indicated that items could be solved with two or more developmentally appropriate strategies. In some cases, teachers and mathematicians shared four unique strategies. As one example, a mathematician shared numerous strategies for an eighth-grade functions item that spanned different representations (i.e., symbols, graphs, and tables), as well as different procedures using those representations. Items were also complex enough such that teachers believed potential test takers would need time to reflect on a viable solution strategy pathway. In many cases, teachers emphasized the need for respondents to reflect on the problem's situational context, then connect ideas to mathematical strategies, and ultimately act on a strategy that had potential to arrive at a solution. Realism was often discussed among panel members. During later stages of data collection, items were deemed to need further revisions because some contexts may be real to one group of students (e.g., a cell phone bill) but not others. Some students shared that a cell phone plan may have limited minutes but that many students might not understand specific components of a cell phone bill (e.g., overage). For instance, Elias was a multilingual student from an urban Mountain West district. He shared during a think aloud that “I don’t think most students understand that minutes cost money. Most students don’t pay their own cell phone bill. Also, many people just have unlimited minutes like on the commercials.” Items like this one were revised and resent to panel members for review to confirm that changes were adequate. In this case, the cell phone bill item was revised to focus more on the quantity of cell phone minutes used rather than connecting cell phone minutes used and cost of a cell phone.

Claim #3: Limited bias

Some initial items were flagged for bias for reasons involving topics like specialized knowledge of different sports (e.g., free throw in basketball) or phrasing in the item that suggested a student had cultural experience with a context (e.g., you went to the beach.) Feedback led to revisions such that the team handled it and resent items to the panels. As an example, the free throw item was revised to focus on throwing a basketball into a hoop. Item revisions helped to orient tasks like this one to general sports ideas that were also located in K-8 physical or health education standards. Items that used “you” were revised to include an individual’s names. Care was taken to be culturally relevant; using names suggested by students from different geographic regions. Thus, it was more likely that students could perceive peers in their problems when they saw names relevant to their local culture. Revised items that were deemed limited in bias were shared with potential respondents during the think alouds. Students confirmed that they felt items contained no observed bias. Our team does not believe there is zero bias across the items but rather, it is limited in scope and not detected by the scholars, practitioners, and potential respondents involved in this study.

Discussion and Next Steps

Our goal was to explore the degree to which evidence supports validity claims of the CAT PSM 6-8. The a priori claims approach adheres to modern standards and best practices in assessment development (AERA et al., 2014; Folger et al., 2023). Based on the evidence and claims presented in this proceeding, test users may feel confident knowing the CAT PSM 6-8 does what it intends. Far too many tests provide insufficient information for test users (Bostic et al., 2022), which can lead to issues including but not limited to (a) spurious findings, (b) negative implications for test takers, and (c) less instructional time for K-12 students (AERA et al., 2014; Cronbach, 1988).

Bias is something that cannot be eliminated (AERA et al, 2014) and instead, is intended to be limited and balanced across an item set. In the case of these items, teachers, bias panel members, other scholars, and students did not perceive negative bias in the final versions of the items; however, that does not necessarily mean there is no bias. Instead, it provides strong qualitative evidence for a lack of observed bias. Further quantitative analyses will be performed using differential item functioning (DIF) to explore whether items grouped onto the CAT PSM have unbalanced bias, which will take place after test administration in May 2023.

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Reframing STEM and Preservice Education

EMBEDDING CLINICAL EXPERIENCES WITHIN A MATHEMATICS METHODS COURSE

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A primary concern within teacher preparation is the disparity between teaching practices learned in programs and teaching practices enacted in typical P-12 classrooms. One model that addresses this disconnect is site-based, embedded experiences during methods courses. Through a collective focus on mathematics knowledge, pedagogy, partnerships, and clinical settings, we present two case models that engage in this critical work showcasing different parameters within elementary mathematics methods courses. Preliminary findings suggest these intentional clinically-centered methods experiences are beneficial in connecting theory to practice.

Introduction and Literature Review

The disconnect between teaching practices bolstered within teacher preparation programs and the teaching practices enacted in typical classrooms is often noted as one of the primary concerns within teacher education literature and policy documents (American Association of Colleges for Teacher Education [AACTE], 2018; Ronfeldt, 2021). While teacher candidates gain valuable theoretical knowledge during their formal preparation programs, research notes teacher candidates are provided limited opportunities to enact this theoretical knowledge within authentic classroom contexts (e.g., Grossman, 2009). To address the disconnect between teacher preparation programs and classroom contexts, policy and reform documents foreground clinically-centered approaches focused on partnerships (e.g., AACTE, 2018; Association of Mathematics Teacher Educators [AMTE], 2017; National Council for Accreditation of Teacher Education [NCATE], 2010; Ronfeldt, 2021). For instance, NCATE (2010) recommends the "establishment of strategic partnerships for powerful clinical preparation" and "the dynamic integration of clinical preparation throughout every facet of teacher education" (p. 5). The AMTE's *Standards for Preparing Teachers of Mathematics* (2017) also promotes the importance of clinical settings (Standard P.5) and collaborative partnerships (Standard P.1) as critical components of effective preparation programs of mathematics teachers.

The importance of clinically-centered teacher preparation noted within policy documents is echoed in research (e.g., Darling-Hammond, 2014; Dunst et al., 2020). Strong clinical experiences can provide the opportunity for teacher candidates to abandon misconceptions and beliefs by being confronted with congruent messages between teacher preparation and classroom instruction (Dunst et al., 2020). Additionally, Grossman and colleagues' (2011) concluded that intentional clinical experiences have positive implications on instructional decisions as teacher candidates move into student teaching and then into their first teaching positions. Clinically-centered partnerships are also necessary in preparing and supporting teacher candidates' pedagogical development for teaching a diverse student population (Rowan et al., 2021).

With the need for more clinically-centered preparation programs, scholars and practitioners have turned to methods courses to strengthen the connection between the theoretical knowledge and authentic classroom contexts. Research indicates that the integration of clinical experiences within methods courses is beneficial for teacher candidates' development (Darling-Hammond, 2014; Grossman, 2021) and that methods courses support teachers' perception of their readiness to teach and persistence in the profession (Ronfeldt et al., 2013). As such, embedded, school-based mathematics methods courses are one option for connecting mathematics content, pedagogy, and classroom practices. Within embedded methods course experiences, teacher candidates are provided immersive learning opportunities within authentic classroom settings, bridging the disconnect between school-and university-based teaching and learning through "theoretically sound and practically relevant" experiences (Hodges & Mills, 2014, p. 249).

Theoretical Framework

The theoretical framework used for conceptualizing embedded school clinical experience was Oonk, et al.'s (2015) concept of theory-enriched practical knowledge. Theory-enriched practical knowledge leverages the teacher educator's intentional support and reflective dialogue to help teacher candidates see the connection between theory and practice, aligning the language and expectations between teacher preparation and classroom practices (Hodges et al., 2017; Oonk et al., 2015). By creating specific, regularly occurring clinically-centered experiences, teacher candidates can make meaning of theoretical concepts and apply theories. This study was guided by following research question: How do clinically-centered experiences influence teacher candidates' theory-enriched practical knowledge?

Models for Site-Based, Embedded Clinical Experiences

We present two case models that both engage in this critical work, while doing so with different parameters.

5-Week Experience

This case includes a partnership between one elementary school and one elementary mathematics methods course. The elementary school principal and the mathematics teacher educator (MTE) created a clinical experience that included teacher candidates in grade-level professional learning communities (PLCs) and co-teaching with small group instruction based on assessment cycle data (Karathanos-Aguilar & Ervin-Kassab, 2022). The elementary school employed a co-teaching model for instruction, pairing four to five teacher candidates with two experienced elementary teachers who served as mentors. Each elementary classroom served around 40 students. Teacher candidates would be at the school for both the teachers' planning time and their mathematics teaching. This planning time was essential, allowing them the time and space to plan with mentors and peers, assess student work, and plan interventions and small group instruction. The candidates also used the planning time to confer with their MTE who held workspace at the school. Teacher candidates were responsible for interventions or small group instruction after analyzing formative assessment data, identifying students' learning gaps, and planning targeted activities. This process of planning, teaching, and analyzing helped teacher candidates become focused on a specific goal for student learning.

Full-Semester Experience

This case takes place in an embedded, site-based elementary mathematics methods course (e.g., Hodges & Mills, 2014; Hodges et al., 2017). The course is taught in a local professional development school (PDS) rather than on campus to allow engagement with elementary students during each class session. For this particular semester, the MTE partnered with a third-grade teacher and her students. Generally, the methods class begins with the MTE and candidates meeting separately in their own classroom to reflect on readings, solve mathematics tasks, and experiment with instructional materials. This is followed by a period of "setting the stage" for the upcoming work with third-grade students, such as exploring a learning progression, learning about student invented strategies, and learning any relevant content or pedagogical content knowledge needed during the lesson. Behind the scenes and prior to the class session, the MTE and third-grade teacher collaboratively set the learning goal, plan the instructional time and activities, and decide who will lead what portion of the time (e.g., third-grade teacher or MTE modeling as well as teacher candidate-student(s) pairings/grouping). Teacher candidates then enter the third-grade classroom for observations of the classroom teacher and/or MTE. The teacher candidates are partnered with one or more students throughout the semester during the classroom interactions where they look closely and listen carefully to the things the third-graders say and do as they engage with mathematical tasks, while the third-grade teacher and MTE provide in-the-moment coaching supports. After the time with third-graders, teacher candidates, the MTE, and classroom teacher (if available) return to their separate classroom to debrief and make meaning of the data collected during the lesson. This collaborative debriefing and reflecting allows for growth from everyone involved and provides the space and time to determine appropriate and responsive future teaching.

Methods

We used a case study approach to understand the influence of two distinct embedded mathematics methods experiences. Qualitative data was gathered from student reflections. A grounded theory approach was used to analyze the data with a constant comparative method (Glasser, 1978). Codes based on the theoretical framework were anticipated and used as starter codes, while others emerged from the data (Miles et al., 2020). Potential a priori codes included: effective practice, student engagement, authentic learning, and reflection. Induction, deduction, and verification were used to identify the themes and then solidify the themes with verification across the data (Glasser, 1978). The researchers reached consensus by classifying the codes into common themes collaboratively.

Findings

Preliminary findings display positive outcomes of intentional, embedded clinical experiences grounded in theory-enriched practical knowledge (Oonk et al., 2015). Initial themes within each of the two clinically-centered methods course experiences are presented below.

5-Week Experience

Teacher candidates found embedded clinical experiences to be valuable according to the reflection data (i.e., a weekly log). Even the teacher candidates that didn't verbalize a benefit, grew in their focus and increased in theory-enriched practical knowledge of assessing, goal setting, and planning as noted in the reflections. The experience allowed for the marrying of theory to practice especially in terms of fostering discourse and the art of questioning. One teacher candidate noted this in her weekly log:

> I was satisfied with the way that the teachers were asking the students to explain how they got their answer. The teachers would ask higher order thinking questions to really

> get the students thinking and it showed. The teachers really let the students try to figure everything out on their own…The satisfying part is watching the students start to understand a challenging problem as they work through it.

Another benefit was seeing and understanding the distinction between engagement and compliance. A teacher candidate noted this in her weekly log:

> I enjoyed watching students learn through a hands-on experience. They worked through the lesson with Mrs. Matson [classroom-based educator] by folding their papers into various equal parts. I could tell that students enjoyed this lesson and had a meaningful learning experience because they were engaged.

Another teacher candidate said this about the same lesson:

> I think the most satisfying experience was seeing how Mrs. Matson allowed the students to experiment with their folding. Letting go of that control and allowing students to take as much time as they needed, I feel is a very valuable piece of math education. Students should be able to experiment with manipulatives.

This lesson was one that was connected to the methods course materials and the teacher candidates were able to see how the teaching of mathematics could and should be student centered.

There were three teacher candidates remained in a mindset of just doing a task and not internalizing the impact of their decisions (Fuller, 1969). These teacher candidates wrote one to two sentences for their weekly log and did not respond to the feedback from their MTE. An additional preliminary finding for the teacher candidates in the five-week experience was that these teacher candidates do not have enough time and experience in the mathematics classroom.

Full-Semester Experience

Similar to those within the 5-week experience, teacher candidates immersed in a full-semester site-based, embedded methods course indicated the value of the experience. Paralleling the ideas put forth by Oonk and colleagues (2015), teacher candidates (whom we refer to as tall teachers) indicated the embedded elementary mathematics method course and weekly opportunities to work with their third-grade students (whom we refer to as small teachers) was a way to make sense of theoretical knowledge in practical ways, noting "theory can be so consuming so it's neat to see it in action." Additionally, teacher candidates saw the experience as a space for collective learning, "While working with small teachers, I am able to learn from them. They are teaching me." Teacher candidates experienced the connection from theory to practice as we learned theory together and put it into action immediately with our partner classroom. One teacher candidate indicated that "working with children creates real scenarios and in the moment learning…this is the best experience for me to learn."

Teacher candidates also reported on how the opportunities provided within an embedded course allowed their confidence to grow, both as mathematicians themselves and teachers of mathematics. One teacher candidate reflected on how the experience "reminded me of what it's like to be confident in my answers and that math lessons shouldn't be so overwhelmingly frightening like how I used to struggle in silence and not ask questions." Other teacher candidates indicated that working with small teachers as part of our course "impacted the way they view teaching in general," and that "the opportunity to see students engaged and enjoy math makes me less weary to teach it."

Many of their reflections centered on effective practices related to the teaching and learning of mathematics. For instance, teacher candidates were able to learn to elicit student thinking, pose purposeful questions, and understand multiple solution paths within problem solving. Teacher candidates indicated how they "had a chance to see the strategies they [small teachers] choose to apply to solve a problem," "see the thinking behind kids answers," "learn how to elicit thinking without giving them [small teachers] answers," and "love wondering with my students and seeing a lightbulb go off in their head." As these reflections show, teacher candidates were focused on their small teachers' thinking and reasoning and learning ways they can allow students' ownership in their learning, which can often be difficult to do without ample opportunities to engage in this type of teaching. The course also allowed teacher candidates to see students as capable doers of mathematics, with one teacher candidate noting "[working with my small teacher] made me realize that I need to give kids more credit. They can figure out things if we give them the opportunity to."

Teacher candidates also considered how the learning taking place within the embedded methods course transferred to their teaching and learning in other settings. One teacher candidate reflected on how the embedded course "has made it easier when working with students at my internship school." However, the transfer took time and persistence for some, with one teacher candidate noting "I am still trying to incorporate some things in my home school," indicating that a possible disconnect may persist or at least more time is needed for confidence in transferring their learning when moving into different contexts outside of the methods course setting.

Discussion

Although we engaged in this clinically-centered approach to methods courses differently, similar initial themes emerged from our teacher candidates' reflections and work that impacted their theory-enriched practical knowledge (Oonk et al., 2015). Teacher candidates across both embedded experiences indicated the value of the authentic learning opportunities and reflected on how these experiences created a bridge between theory and practice. Teacher candidates were able to engage in opportunities where elementary students' thinking and reasoning were positioned at the forefront of instruction, witness the brilliance of students' mathematical knowledge when they are given the authority to share, and explore meaningful and engaging mathematics learning opportunities. While many of the teacher reflections were centered around the positive aspects of the embedded experiences, a few considerations are important to note. Teacher candidates indicated that more time in authentic settings is needed and that the transfer of effective practices into different contexts (e.g., internship classrooms) may need more attention. Schools can reap the benefits of opening their schools and classrooms to teacher preparation programs (Ronfeldt, 2015) like these two models presented here. As we move forward in our embedded work, these are important points of reflection from our teacher candidates to keep in mind. How might we provide more time and space for teacher candidates to continue to bridge the theory to practice divide both within our courses and across programmatic settings? How do we increase the likelihood of transfer within new contexts (e.g. student teaching)? Intentional, collaborative clinically-centered partnerships with reflective mentorship aligned with coursework are essential in preparing teacher candidates with strong, high-quality pedagogy (Dunst et al., 2020). We presented two case models that both engage in this critical work while doing so with different parameters. We know other models exist. In order to support the mathematics learning and pedagogy of teacher candidates, there must be some changes within teacher preparation to increase theory-enriched practical knowledge (Oonk et al., 2015) and provide experiences where this knowledge can develop and flourish.

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SUPPORTING THE STEM PATHWAY AT APPALACHIAN STATE

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Numerous national, state, and institutional studies suggest that the corequisite mathematics model accelerates students' completion of their introductory college-level mathematics course in their first year of college. Current peer-reviewed literature, while positive, overwhelmingly lacks replicable details for institutions wanting to implement the corequisite mathematics model. Our work seeks to fill this gap.

Evidence suggests that the use of traditional developmental education classes in mathematics is ineffective and recent years have seen improvements and changes to the traditional development education model (Howell et al., 2023). One such change with resounding effectiveness is implementing the corequisite mathematics model, where students concurrently enroll in college-level and developmental mathematics coursework for just-in-time academic support (Ryu et al., 2022). Many national, state, and institutional studies suggest that the corequisite mathematics model eliminates exit points and improves college-level mathematics coursework completion rates (Logue et al., 2019; Ran & Lin, 2019).

With such promising evidence, states, systems, and institutions of higher education nationwide have quickly adopted the corequisite mathematics model to accelerate students' completion of their introductory college-level mathematics course in their first year of college. This broad-scale adoption and ongoing evidence continue to reinforce the belief that the corequisite mathematics model works to help accelerate students' completion of their first college-level mathematics course, yet what is less known are the structural factors of the corequisite mathematics model that provide the greatest impact on student success (Howell et al., 2023). This gap in the literature leaves many unanswered questions for those looking to implement a corequisite mathematics model.

Our research efforts are designed to keep the STEM (Science, Technology, Engineering, and Mathematics) pathway as a viable degree pathway for interested students by ensuring access to calculus and thus facilitating timely degree completion. Our corequisite mathematics courses are the means by which students receive support for the calculus sequence and serve as the focus of this study. We share the design and implementation of our corequisite courses along with preliminary quantitative and qualitative findings to inform the following research questions:

1. What characteristics of the corequisite mathematics courses promote student success, as defined by the final course grade in the target courses in the calculus sequence?
2. Do students at risk of not completing the target courses in the calculus sequence with grades of C or better successfully complete the target courses at a higher rate when they are enrolled in the corequisite course(s)?

Literature Review

A recent meta-analysis of the literature on corequisite support courses (Howell et al., 2023) revealed a lack of research on corequisite courses for calculus, on the specific structuring of effective corequisites, and on placement for both the corequisite and target courses. Several common themes emerged from the available literature and focused on mathematics pathways, student placement, and corequisite course design.

Mathematics Pathways

Implementing corequisite mathematics courses at higher education campuses commonly coincides with mathematics pathway reforms. Mathematics pathways reform is the intentional effort of higher educational campuses to align and enroll students in the right mathematics course for their program of study. As a part of mathematics pathways efforts, institutions provide at least two "math pathways" for students; the sequence of precalculus and calculus classes for those majoring in a STEM field is often referred to as the "STEM Pathway" while pathways for students with other majors might be referred to as the "Quantitative Reasoning Pathway" or "Statistics Pathway." While corequisite mathematics courses and math pathways reform are commonly implemented together, few articles meaningfully examine the connections between the two efforts and their impact on student success (Howell et al., 2023). Of those articles that examine both corequisite mathematics and mathematics pathways reform, Ran and Lin (2019) offered significant results in their study of deidentified state administrative data for 52,036 first-time-in-college students who entered one of thirteen community colleges in Tennessee between 2010-2011 and 2016-2017. Ran and Lin (2019) found that students placed in corequisite mathematics courses were up to 18 percentage points more likely to pass their introductory, college-level mathematics course by the end of their first year in college.

In another study that examined the connection between corequisite mathematics courses and mathematics pathways reform, Andrews and Tolman (2021) examined a population of 1,934 students who enrolled in at least one corequisite English or mathematics course at a community college in the southeastern United States between 2015 and 2018. They found that students who enrolled in corequisite remediation and an appropriate mathematics course for their academic major were more likely to be academically successful.

Other articles examine the effectiveness of corequisite mathematics courses within the context of math pathways reform, but they do not make explicit the connection between the two efforts and observed student success (e.g., Childers et al., 2021; Logue et al., 2019; Wakefield, 2020). These articles do note more positive outcomes for student success for those enrolled in corequisite mathematics courses than those in a traditional developmental mathematics model.

Student Placement

In the literature about corequisite mathematics classes, student placement is another common theme, though the placement criteria are as varied as the institutions implementing them (Howell et al., 2023). Broad placement criteria loosely identified in the articles include high school grade point average (GPA), standardized assessment tests generated by for-profit companies (e.g., SAT, ALEKS, Accuplacer), and locally generated in-house tests from the mathematics department at an institution (Andrews, 2021; Beamer, 2021; Wakefield, 2020). Overwhelmingly though, replicable details of placement criteria are missing from the literature.

Corequisite Course Design

Corequisite mathematics courses may address the mathematics content traditionally taught in stand-alone developmental-education mathematics courses (Boatman, 2021; Wakefield, 2020), or they may be designed to provide just-in-time academic support aligned to their corresponding introductory, college-level mathematics course. The literature about corequisite mathematics models describes several common themes, including faculty experience teaching corequisite mathematics courses, use of online, self-paced curricular materials, use of non-standard pedagogical practices, and incorporation of student success skills such as self-regulation (Howell et al., 2023).

As noted above regarding placement criteria, there is great variability in how these themes might be addressed at different institutions. For example, some articles indicated that their corequisite mathematics courses are taught by faculty with experience in the corresponding target course (e.g., Beamer, 2021; Childers et al., 2021; Hancock et al., 2021) while others described instructors as graduate students, undergraduate students, or faculty without experience in the corresponding mathematics course. Buckles et al. (2019) and Wakefield (2020) described corequisite courses that use online, self-paced curricular materials and did not describe a course instructor. Other design characteristics for corequisite mathematics models, such as total number of contact hours, student assessment methods, faculty preparation, curricular alignment between the corequisite course and target course, and measures of student success also differ greatly across the literature.

Methodology

Our study takes place at Appalachian State University, a moderately sized university in the University of North Carolina System (https://www.northcarolina.edu/about-us/). Appalachian has a total enrollment of approximately 20,000 students, a student-to-faculty ratio of 16:1, and classes with an average size of 25 students (https://www.appstate.edu/about/facts/). The class size for the target class of this study, Calculus 1, varies between 33 to 36 students, and the corequisite class size is capped at 32 students.

The UNC System Math Pathways Project

As part of the UNC System Math Pathways Project that began in 2017, each of the 17 institutions in the system signed a letter of commitment to implement various actions that would support student success in mathematics. All 17 institutions agreed to eight core elements of participation, which included designing and implementing at least two math pathways for students as well as encouraging all students to complete the first math course required for their program of study within their first 30 hours of enrollment. Beyond the eight core elements, there were numerous “institutional-selective” options that institutions could also pursue as a part of their participation in the UNC System Math Pathways Project (https://www.northcarolina.edu/impact/system-wide-initiatives/math-pathways/).

Implementing Math Pathways at Appalachian State University

This study focuses on efforts Appalachian implemented to address the core Math Pathways recommendation that students complete their first college-level mathematics requirement in their first 30 hours of enrollment. Appalachian already had clear mathematics pathways for students majoring in STEM fields and for students majoring in other areas. The STEM Pathway is the more involved pathway and has higher failure rates than the other pathways Appalachian offers. For our STEM majors, Calculus 1 is the first college-level mathematics requirement that counts towards their degree, since Precalculus does not count towards their general education requirements. Thus, the goal of the recommendation as it pertains to STEM majors is to have those students complete Calculus 1 in their first year on campus. To meet this goal, Appalachian focused on better placement methods and also developed corequisite classes.

Placement

There are two department-made placement exams; the Math Placement Test is used to determine whether a student needs to complete developmental mathematics or not and the Calculus Readiness Test determines whether a student is ready for calculus. A student could enroll in Calculus 1 by having credit for a college-level precalculus course, earning a 3 or higher on the AP Calculus test, passing the Calculus Readiness Test, or completing 85% of a self-paced ALEKS prep for calculus course. When the corequisite class first began, any student could self-select into the course. There were instances of high-performing calculus students who chose to enroll in the corequisite support course, so the course was not necessarily attracting the students for whom it was intended. Starting in Fall 2022, students who missed the cutoff score on the Calculus Readiness Test by 1 point, and chose not to take the self-paced ALEKS course, were given the option of either taking Calculus 1 with the corequisite support course or Precalculus. This provides many students with an option to complete their mathematics requirement in one semester, rather than two.

Corequisite Course Design

There are several factors considered when designing the corequisite support course: the scheduling of the course, the faculty who teach the course, and the content of the course. Appalachian's Calculus 1 corequisite course meets for two hours per week and is offered at times that do not conflict with any of the Calculus 1 sections since students from any section of Calculus 1 can enroll in the corequisite course. It is taught by faculty who have experience teaching the target course, Calculus 1, and who are also engaged in the scholarship of teaching and learning. Appalachian usually offers one Calculus 1 corequisite course each semester, so the instructor's knowledge of the Calculus 1 course is vital as is the collaboration Appalachian fosters between colleagues that allows the instructor to check in with any of the Calculus 1 instructors as needed. Since students in the corequisite course are from multiple sections of Calculus 1, the corequisite course must be designed in such a way that it meets the needs of all the students. To meet this goal, the corequisite instructors agreed that in order to truly support success in Calculus 1, the corequisite course should not place additional time burdens on students by requiring lengthy homework assignments or study time for quizzes or tests for the support course. The content of the corequisite course is determined by "back mapping"; that is, paying careful attention to specific areas in Calculus 1 that traditionally are more difficult for students and then identifying prerequisite content that needs to be addressed in the corequisite course to offset the anticipated difficulty. Corequisite faculty feel very strongly that teaching methods focus on understanding, adopting an adaptation of Van de Walle's (2014) definition of understanding: *Understanding is a measure of the quantity and quality of connections that you have to an idea.* Thus, instructors design tasks that highlight important mathematical features, elicit conversations about vocabulary, allow for comparing and contrasting, and build connections between representations. Instructor collaboration supports the implementation of active learning experiences (Freeman et al., 2014).

Discussion and Findings

The corequisite and support course effort at Appalachian began with a single corequisite course for Calculus 1 in 2015. Over the next five years, the support efforts grew to include revisions of the placement procedures to the structure described above and corequisite courses for Precalculus and later, Calculus 2. The pandemic of 2020 caused significant disruption to the efforts to study the impact of these support courses as well as to the continued implementation of these courses in the way they were designed. Thus, the data analysis is considered preliminary despite the seven-year history of implementation. The initial results from Calculus 1 and its corequisite are promising, as shown in Table 1.

Table 1

D, F, or Withdraw Rates for Calculus 1 Students vs. Those Who Co-enrolled in the Corequisite

Spring 2018 - Fall 2019		Fall 2022	
All Calculus	With Corequisite	All Calculus	With Corequisite
33.8%	21.3%	37.9%	17.6%

With regard to our first research question, we have informal student feedback that identifies the connection formed with the corequisite course instructor and the focus on teaching for understanding in the corequisite course as beneficial to their learning. "Presenting questions in different ways", "more visual learning", and "deeper explanations" are seen by students as benefits of participating in the Calculus 1 corequisite. As with others studying the characteristics of corequisite courses and their impact on student success, we are unable to claim that our instructional methods in the corequisite class are the cause of higher pass rates for these students in Calculus 1, though we feel strongly that this is the case.

Likewise, for our second research question, our preliminary data show that students enrolled in the corequisite course for Calculus 1 have lower DFW rates than the overall Calculus 1 DFW rates. However, we have not been able to conduct a formal comparison between students at risk of not completing the target courses in the calculus sequence with grades of C or better who do or do not enroll in the corequisite course; our efforts have been focused on enrolling all of these students in the corequisite course. A review of the student data from Fall 2022, the first semester after we changed our placement requirements, indicates that 58% of the students enrolled in the corequisite course missed the Calculus Readiness Test cutoff score by one point.

As more students participate in the Calculus 1 corequisite course, a more rigorous statistical analysis can be performed. The creation of a survey tool and implementation of techniques such as propensity score matching could be utilized to gauge impact despite the confounding issue of student self-selection into the course. Also, a more qualitative approach to look at student characteristics would help determine the corequisite's impact on different student populations (e.g., minoritized populations, females, first-generation students, and transfer students).

Conclusion

To aid students in successfully completing their introductory college-level mathematics course, many colleges and universities have adopted corequisite mathematics models. The available peer-reviewed literature on these models indicates promising impacts, but the diversity of the implemented models yields no clear format that other institutions can easily replicate (Howell et al., 2023). To help fill this gap in the literature, we have provided details of our corequisite model at Appalachian State, including information about our placement procedures, corequisite course design, and a description of our successes and challenges. We continue to support the use of the corequisite mathematics model instead of traditional prerequisite mathematics courses.

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